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CRITICAL POINTS OF ONE PARAMETER FAMILIES OF MAPS OF THE INTERVAL

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ABSTRACT. It is shown that some of the periodic phenomena which is well known to occur for the critical point of the quadratic family $f_s(x) = sx(1-x)$ (and other C^1 families with a single critical point) occurs for each critical point in C^1 families with an arbitrary (possibly infinite) number of critical points. Also, some of the same behavior occurs in families of maps (which are not necessarily differentiable) where a critical point has derivative zero on either the left or the right side. A stronger condition is obtained when the derivative on the right is zero.

This paper is concerned with the periodic behavior of critical points of one parameter families of maps, f_s, of an interval to itself. We let $(Df_s)^+(x)$ (resp. $(Df_s)^-(x)$) denote the derivative of f_s at x on the right (resp. left) and $(f_s)'(x)$ denote the derivative of f_s at x. We let $C^1([0,1],[0,1])$ denote the space of continuously differentiable maps from $[0,1]$ to itself with the C^1 (uniform) topology. We will use the term period to mean least period [**1**].

THEOREM. *Let $F:[a,b]\times[0,1]\to[0,1]$ be continuous and let $f_s(x) = F(s,x)$. Suppose that for each $s\in[a,b]$ there is a point z_s in the open interval $(0,1)$ such that the map $s\to z_s$ is continuous, and $f_a(z_a)\leqslant z_a$ while $f_b(z_b)=1$. Suppose also that for each $s\in[a,b]$, $f_s(0)=f_s(1)=0$.*

(1) *Suppose $(Df_s)^+(z_s)=0$ for each $s\in[a,b]$, and there is a constant $\gamma>0$ such that $(f_s)'(x)$ exists for all $x\in(z_s, z_s+\gamma)$, and $(Df_s)^+(x)$ varies continuously with s and x for $s\in[a,b]$ and $x\in[z_s,z_s+\gamma)$. Then there is a sequence (s_n) in (a,b) with $s_1<s_2<s_3<s_4<s_5<\cdots$ such that for each $n=1,2,3,\ldots$, z_{s_n} is a periodic point of f_{s_n} of period n.*

(2) *Suppose $(Df_s)^-(z_s)=0$ for each $s\in[a,b]$, and there is a constant $\gamma>0$ such that $(f_s)'(x)$ exists for all $x\in(z_s-\gamma,z_s)$, and $(Df_s)^-(x)$ varies continuously with s and x for $s\in[a,b]$ and $x\in(z_s-\gamma,z_s]$. Then there is a positive integer N and an element s_n of (a,b) for every integer n with $n=1$ or $n\geqslant N$ such that z_{s_n} is a periodic point of f_{s_n} of period n, and $s_1<s_N<s_{N+1}<s_{N+2}<\cdots$.*

(3) *Suppose that for each $s\in[a,b]$, $f_s\in C^1([0,1],[0,1])$, and the map $s\to f_s$ from $[a,b]$ to $C^1([0,1],[0,1])$ is continuous. Suppose also that $(f_s)'(z_s)=0$ for all s. Then there is a sequence (s_n) as in (1) and also a sequence (t_n) in the interval $[s_2,s_3)$ with $s_2=t_1<t_2<t_3<\cdots$ such that for each $n=1,2,3,\ldots$, z_{t_n} is a periodic point of f_{t_n} of period 2^n.*

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In (3) we have a C^1-continuous family of maps f_s with a family of critical points z_s (which vary continuously with s), and the point $(z_s, f_s(z_s))$ moves from below the diagonal to a height of 1. The Theorem asserts that z_s becomes periodic of period 1, 2, 4, 8, 16, 32, ..., 3, 4, 5, ... (in succession). The standard example, $f_s(x) = sx(1-x)$ where $0 \leqslant s \leqslant 4$, satisfies the hypothesis of (3) with $z_s = \frac{1}{2}$. Of course the conclusion of (3) is well known for this example and for similar examples with a single critical point, and, in fact, much more detailed (and quite beautiful) information is known [**1**]. In these examples the kneading theory [**3**] determines the dynamics (to a large extent), and the first time z_s becomes periodic it is fixed (i.e. has period 1), the next time z_s is periodic it has period 2, then period 4, etc. The Theorem given here does not assert this property, and in fact this property does not hold for the more general types of examples to which the Theorem applies. For example, the family

$$f_s(x) = 64(s+1)^{-2}x(1-x)\big((x-1/2)^2 + s/4\big), \qquad 0 \leqslant s \leqslant 1,$$

satisfies the hypothesis of (3) with $z_s = \frac{1}{2}$. The Theorem asserts that z_s becomes periodic of period 1, 2, 4, 8, 16, 32, ..., 3, 4, 5, ... in succession, but z_s may become periodic of other periods before 1, between 1 and 2, etc. In this example, it is easy to verify that before becoming fixed, z_s actually does become periodic of other periods.

Note that for a given family of maps there may be several (or infinitely many) families of critical points z_s satisfying the hypothesis of (3). For example, the family

$$f_s(x) = 16sx(1-x)\big((x-1/2)^2 + s/4\big), \qquad 0 \leqslant s \leqslant 1,$$

satisfies the hypothesis of (3), where we may take $z_s = \frac{1}{2} - \sqrt{(1-s)/8}$, $z_s = \frac{1}{2}$, or $z_s = \frac{1}{2} + \sqrt{(1-s)/8}$. Another example is the family f_s defined for $0 \leqslant s \leqslant 1$ by

$$f_s(x) = \begin{cases} 16s\big(-x^2 + \frac{1}{2}x\big) & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ s & \text{if } \frac{1}{4} \leqslant x \leqslant \frac{3}{4}, \\ 16s\big(-x^2 + \frac{3}{2}x - \frac{1}{2}\big) & \text{if } \frac{3}{4} \leqslant x \leqslant 1. \end{cases}$$

Here, we may take z_s to be any continuous family of points in the interval $[\frac{1}{4}, \frac{3}{4}]$.

Note that in (3), when z_s is periodic, it is automatically stable since $(f_s)'(z_s) = 0$ [**1**].

An example of a family satisfying (1) is given by

$$f_s(x) = \begin{cases} sx/2 & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ sx(1-x) & \text{if } \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

for $0 \leqslant s \leqslant 4$. A similar example is given by

$$f_s(x) = \begin{cases} (s/4)\big(1 - \sqrt{1-2x}\big) & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ sx(1-x) & \text{if } \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

for $0 \leqslant s \leqslant 4$. In both examples, $z_s = \frac{1}{2}$.

An example of a family satisfying (2) is given by

$$f_s(x) = \begin{cases} sx(1-x) & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ s(1-x)/2 & \text{if } \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

for $0 \leqslant s \leqslant 4$. In this example one can verify that the critical point $z_s = \frac{1}{2}$ is never a periodic point of f_s of period 2. Thus, the conclusion of (1) does not hold with the hypothesis of (2).

We remark that some examples similar to ones described here were studied numerically in [**2**].

We now proceed to prove the Theorem.

PROOF OF (1). By hypothesis, $f_a(z_a) \leqslant z_a$ and $f_b(z_b) > z_b$. Hence, by continuity of F and z_s, for some $s_1 \in (a, b)$, $f_{s_1}(z_{s_1}) = z_{s_1}$. We may takes s_1 to be the maximal element of (a, b) with $f_{s_1}(z_{s_1}) = z_{s_1}$. Then for all $s > s_1, f_s(z_s) > z_s$.

Since $(Df_{s_1})^+(z_{s_1}) = 0$ and $(Df_s)^+(x)$ varies continuously with s and x for $x \in [z_s, z_s + \gamma)$, there is an open interval $N_1 \subset (a, b)$ with $s_1 \in N_1$ and a $\delta > 0$ with $\delta < \gamma$, such that $|(Df_s)^+(x)| < 1$ for all $s \in N_1$ and $x \in [z_s, z_s + \delta)$. Since $(f_s)'(x)$ exists for $x \in (z_s, z_s + \delta)$, $|(f_s)'(x)| < 1$ for $s \in N_1$ and $x \in (z_s, z_s + \delta)$. It follows from the continuity of F and the fact that $f_{s_1}(z_{s_1}) = z_{s_1}$, while $f_s(z_s) > z_s$ for all $s > s_1$, that there is an open interval $N_2 \subset N_1$ with $s_1 \in N_2$ such that $f_s(z_s) \in (z_s, z_s + \delta)$ for all $s \in N_2$ with $s > s_1$.

Let $s_\lambda \in N_2$ with $s_\lambda > s_1$. Then $f_{s_\lambda}(z_{s_\lambda}) > z_{s_\lambda}$ and $f_{s_\lambda}(z_{s_\lambda}) < z_{s_\lambda} + \delta$. Since $|f'(x)| < 1$ for all $x \in (z_{s_\lambda}, f_{s_\lambda}(z_{s_\lambda}))$, it follows from the mean value theorem that $(f_{s_\lambda})^2(z_{s_\lambda}) = f_{s_\lambda}(f_{s_\lambda}(z_{s_\lambda})) > z_{s_\lambda}$. Since $(f_b)^2(z_b) = 0 < z_b$, it follows by continuity of F that for some $s_2 \in (s_\lambda, b)$, $(f_{s_2})^2(z_{s_2}) = z_{s_2}$. Since $s_2 > s_1$, $f_{s_2}(z_{s_2}) \neq z_{s_2}$, so z_{s_2} is a periodic point of f_{s_2} of period 2. We may take s_2 to be the maximal element of (s_1, b) with $(f_{s_2})^2(z_{s_2}) = z_{s_2}$.

Note that $(f_{s_2})^3(z_{s_2}) = f_{s_2}(z_{s_2}) > z_{s_2}$, and $(f_b)^3(z_b) = 0 < z_b$. By continuity of F, for some $s_3 \in (s_2, b)$, $(f_{s_3})^3(z_{s_3}) = z_{s_3}$. By choice of s_1 and s_2, z_{s_3} is a periodic point of f_{s_3} of period 3. We may take s_3 to be the maximal element of (s_2, b) with $(f_{s_3})^3(z_{s_3}) = z_{s_3}$.

Now $(f_{s_3})^4(z_{s_3}) = f_{s_3}(z_{s_3}) > z_{s_3}$, and the conclusion of (1) follows by repeating the argument of the preceding paragraph inductively.

PROOF OF (2). As in the proof of (1) for some $s_1 \in (a, b)$ (which is chosen to be maximal), $f_{s_1}(z_{s_1}) = z_{s_1}$. Then $f_s(z_s) > z_s$ for all $s \in (s_1, b)$.

If for some $s_\lambda \in (s_1, b)$, $(f_{s_\lambda})^2(z_{s_\lambda}) \geqslant z_{s_\lambda}$, then the proof of (1) applies and the conclusion follows. This is true because the hypotheses concerning differentiability were only used in the proof of (1) to produce such a parameter $s_\lambda \in (s_1, b)$. Hence, we may assume that $(f_s)^2(z_s) < z_s$ for all $s \in (s_1, b)$.

Since $(Df_{s_1})^-(z_{s_1}) = 0$ and $(Df_s)^-(x)$ varies continuously with s and x for $x \in (z_s - \gamma, z_s]$, there is an open interval $N_1 \subset (a, b)$ with $s_1 \in N_1$ and a $\delta > 0$ with $\delta < \gamma$ such that if $s \in N_1$ and $x \in (z_s - \delta, z_s)$, then $|(Df_s)^-(x)| < 1$. Since $(f_s)'(x)$ exists for all $x \in (z_s - \delta, z_s)$, $|(f_s)'(x)| < 1$ for $s \in N_1$ and $x \in (z_s - \delta, z_s)$. It follows from the continuity of F and the fact that $(f_{s_1})^2(z_{s_1}) = z_{s_1}$, while $(f_s)^2(z_s) < z_s$ for all $s > s_1$, that there is an open interval $N_2 \subset N_1$ with $s_1 \in N_2$ such that if $s \in N_2$ with $s > s_1$, then $(f_s)^2(z_s) \in (z_s - \delta, z_s)$.

Let $s_\lambda \in N_2$ with $s_\lambda > s_1$. Then $|(f_{s_\lambda})'(x)| < 1$ for all $x \in ((f_{s_\lambda})^2(z_{s_\lambda}), z_{s_\lambda})$. Since $f_{s_\lambda}(z_{s_\lambda}) > z_{s_\lambda}$, it follows from the mean value theorem that

$$(f_{s_\lambda})^3(z_{s_\lambda}) = f_{s_\lambda}\big((f_{s_\lambda})^2(z_{s_\lambda})\big) > (f_{s_\lambda})^2(z_{s_\lambda}).$$

We claim that for some integer $n > 2$, $(f_{s_\lambda})^n(z_{s_\lambda}) \geq z_{s_\lambda}$. To prove this, suppose that for all $n > 2$, $(f_{s_\lambda})^n(z_{s_\lambda}) < z_{s_\lambda}$. It follows from the mean value theorem that

$$(f_{s_\lambda})^2(z_{s_\lambda}) < (f_{s_\lambda})^3(z_{s_\lambda}) < (f_{s_\lambda})^4(z_{s_\lambda}) < \cdots < z_{s_\lambda}.$$

Let y_{s_λ} be the limit of the increasing sequence $(f_{s_\lambda})^n(z_{s_\lambda})$ $(n = 2, 3, 4, \ldots)$. Then $f_{s_\lambda}(y_{s_\lambda}) = y_{s_\lambda}$ and $f_{s_\lambda}(z_{s_\lambda}) > z_{s_\lambda}$ contradicts, by the mean value theorem, the fact that $|(f_{s_\lambda})'(x)| < 1$ for all $x \in (z_{s_\lambda} - \delta, z_{s_\lambda})$. This establishes our claim that for some integer $n > 2$, $(f_{s_\lambda})^n(z_{s_\lambda}) \geq z_{s_\lambda}$. Since $(f_b)^n(z_b) < z_b$, for some $t \in (s_\lambda, b)$, $(f_t)^n(z_t) = z_t$. We let N denote the period of z_t (under f_t) and set $s_N = t$. Note that $N > 1$ since $t > s_1$, and $f_s(z_s) > z_s$ for all $s > s_1$. We may assume, by choosing t larger and N smaller if necessary, that t is the maximal element of (s_1, b) such that z_t is a periodic point of f_t of period k, where $2 \leq k \leq N$. Now, repeating the argument at the end of the proof of (1), we obtain s_n for each integer $n \geq N$ with $s_N < s_{N+1} < s_{N+2} < \cdots$ such that for each $n \geq N$, z_{s_n} is a periodic point of f_{s_n} of period n.

Proof of (3). Since the hypothesis of (3) is stronger than the hypothesis of (1) we obtain $s_1 < s_2 < s_3 \cdots$ as in the proof of (1).

Note that by the chain rule, $((f_{s_2})^2)'(z_{s_2}) = 0$ and, so, there are neighborhoods N_1 of s_2 in (a, b) and N_2 of z_{s_2} in $(0, 1)$ such that $|((f_s)^2)'(x)| \leq 1$ if $s \in N_1$ and $x \in N_2$. Furthermore, there is a neighborhood N_3 of s_2 with $N_3 \subset N_1$ such that if $s \in N_3$, then $z_s \in N_2$ and $(f_s)^2(z_s) \in N_2$. Let $s_\lambda \in N_3$ with $s_\lambda > s_2$. Then $(f_{s_\lambda})^2(z_{s_\lambda}) < z_{s_\lambda}$ (by choice of s_2) and $[(f_{s_\lambda})^2(z_{s_\lambda}), z_{s_\lambda}] \subset N_2$. Thus, $|((f_{s_\lambda})^2)'(x)| < 1$ for all $x \in [(f_{s_\lambda})^2(z_{s_\lambda}), z_{s_\lambda}]$. By the mean value theorem (applied to $(f_{s_\lambda})^2$),

$$(f_{s_\lambda})^4(z_{s_\lambda}) = (f_{s_\lambda})^2\left((f_{s_\lambda})^2(z_{s_\lambda})\right) < z_{s_\lambda}.$$

Since $(f_{s_3})^4(z_{s_3}) = f_{s_3}(z_{s_3}) > z_{s_3}$, for some t_2 with $s_2 < t_2 < s_3$, $(f_{t_2})^4(z_{t_2}) = z_{t_2}$. By choice of s_2, z_{t_2} is a periodic point of f_{t_2} of period 4. We may take t_2 to be the largest element of (s_2, s_3) such that $(f_{t_2})^4(z_{t_2}) = z_{t_2}$. Then $(f_s)^4(z_s) > z_s$ for all s with $t_2 < s \leq s_3$.

Now, for t slightly larger than t_2, $(f_t)^4(z_t)$ is slightly larger than z_t. Hence, $(f_t)^8(z_t)$ is also larger than z_t (the proof uses the mean value theorem, as above, applied to $(f_t)^4$). Since $(f_{s_3})^8(z_{s_3}) = (f_{s_3})^2(z_{s_3}) < z_{s_3}$ (as $(f_s)^2(z_s) < z_s$ for all $s > s_2$), for some t_3 with $t_2 < t_3 < s_3$, $(f_{t_3})^8(z_{t_3}) = z_{t_3}$. By choice of t_2, z_{t_3} is a periodic point of f_{t_3} of period 8.

The conclusion of (3) follows by repeating the argument of the preceding two paragraphs inductively.

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