

EQUIVARIANT COVERING SPACES AND COHOMOLOGY

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ABSTRACT. The Bredon cohomology of classifying spaces for categories of equivariant covering spaces is considered and shown to correspond to derived functors for the coefficient systems of the Bredon theory.

Introduction. If A is a finite group, then one has the classical result that BA , the classifying space for principal A -bundles, has cohomology equal to the cohomology of A . The purpose of this note is to detect the algebraic invariant which measures the G -equivariant (Bredon) cohomology of equivariant classifying spaces for G a finite group.

Specifically, we consider important categories of G -covering spaces, formulate universal classifying spaces for such objects, and consider their Bredon cohomology. We then realize this cohomology as an algebraic invariant which may be computed via explicit Bar resolutions (Theorem 3.4). These invariants turn out to be derived (Ext) functors associated with the coefficient system of the cohomology.

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1. Equivariant covering spaces. Let A be a finite group. By a (G, A) -covering space $p: E \rightarrow B$, we shall mean a covering space with structure group A such that p is equivariant with respect to given G -actions on E and B . We also require that, if $b \in B$, then the action of $G_b = \{g \in G: gb = b\}$ on $p^{-1}(b)$ is given via some homomorphism $G_b \rightarrow A$.

An important way in which such covering spaces arise is given as follows. Let X be a completely regular left G -space with a right A -action such that the A -action is free and commutes with the G -action (so that $(gx)a = g(xa)$ for each $g \in G$, $x \in X$ and $a \in A$). The orbit space X/A is then automatically a G -space, with the projection $p: X \rightarrow X/A$ G -equivariant. Denote $p(x)$ by $[x]$. Then, for each $x \in X$, $G_{[x]}$ acts on $p^{-1}([x]) \cong A$. Such an identification $\eta: A \rightarrow p^{-1}([x])$ may be obtained by restricting a coordinate function, and two such differ by a translation (that is, if ξ is another, then $(\xi^{-1} \circ \eta)(a) = a_0 a$ for some fixed a_0).

Since the $G_{[x]}$ -action on $p^{-1}([x])$ commutes with the A -action, so does its action on A (via η). Thus, for each $h \in G_{[x]}$, $ha = (he_A)a$, where e_A is the identity in A . The

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assignment $h \rightarrow he_A$ is then a homomorphism $\omega: G_{[x]} \rightarrow A$. Indeed, if h_1 and h_2 are in $G_{[x]}$, then $\omega(h_1h_2) = (h_1h_2)e_A = h_1(h_2e_A) = h_1(e_Ah_2e_A) = (h_1e_A)(h_2e_A)$ since the $G_{[x]}$ - and A -actions commute. One therefore has, for each $h \in G_{[x]}$, $ha = \omega(h)a$ for some homomorphism $\omega: G_{[x]} \rightarrow A$. Since ω depends on the choice of the coordinate function η , denote ω by ω_η . If ξ is another coordinate function, then $\xi^{-1}\eta(e_A) = a_0$, and the action on A determined by ξ is given by $h *_\xi e_A = a_0(h *_\eta e_A)a_0^{-1}$, where $*_\xi$ and $*_\eta$ denote the two actions in question. Thus $\omega_\xi(h) = a_0\omega_\eta(h)a_0^{-1}$, and the two actions are conjugate.

The projection $p: X \rightarrow X/A$ is thus an example of a (G, A) -covering space. This example is, in a sense, the most general example; if F is a left A -space, then the projection $p: X \times_A F \rightarrow X/A$ may be seen to be a (G, A) -covering space, and all (G, A) -covering spaces arise in this way, just as in the nonequivariant case.

If the fixed-sets of the base X/A are connected, then one may select a G -fixed point $[x_0]$ and a fixed identification $\eta_0: A \rightarrow p^{-1}([x_0])$, and use the lifting property to obtain fixed identifications $A \rightarrow p^{-1}([x])$ for all fibers. One is therefore given a fixed homomorphism $\sigma: G \rightarrow A$, such that $G_{[x]}$ acts on $p^{-1}([x])$ via $\sigma|G_{[x]}$ for each $x \in X$. The situation is the same for arbitrary (G, A) -covering spaces $p: E \rightarrow B$ with each B^H connected.

If the fixed-sets of B are disconnected, the action of some isotropy subgroup on two distinct fibers may be unrelated in general. Further, the homomorphisms $G_b \rightarrow A$ need not extend to a G -action. As a consequence, the equivariant classifying space for such objects is complicated, involving as it must all possible actions by subgroups of G on the fiber. We shall therefore restrict attention to the following (simpler) kind of (G, A) -covering space.

DEFINITION 1.1. Let F be a finite discrete space with $A \subset \text{Aut}(F)$, and let $\sigma: G \rightarrow A$ be a fixed homomorphism. Then a $(G, A; \sigma)$ -covering space $p: E \rightarrow B$ is a (G, A) -covering space with fiber F such that, for each $b \in B$, $p^{-1}(b)$ and F are G_b -homomorphic, where G acts on F via σ .

Of course, this definition is meaningful whether or not B has connected fixed-sets.

In [W4] we describe an analogous concept for G -vector bundles, and construct classifying spaces for such bundles. *Mutatis mutandi*, one has the following descriptions of the classifying space for $(G, A; \sigma)$ -coverings.

PROPOSITION 1.2. Let G act on A from the left by the action $g \cdot a = \sigma(g)a\sigma(g)^{-1}$ for $a \in A$. Then BA , with the G -action inherited from the action on A , is the classifying space for $(G, A; \sigma)$ -coverings. \square

Further, if one uses, as one's model for BA , the two-sided geometric Bar construction of [M1], $BA = B(*, A, *)$, then the universal $(G, A; \sigma)$ -covering with fiber F over BA is given by the natural projection $p: B(*, A, F) \rightarrow B(*, A, *)$. Here, G acts on F via σ and on A as above.

Let G and A act, respectively, from the left and right on $EA = B(*, A, A)$ via the action

$$(g, a) \cdot (*[a_n, \dots, a_1]a_0) = *[\sigma(g)a_n\sigma(g)^{-1}, \dots, \sigma(g)a_1\sigma(g)^{-1}]\sigma(g)a_0a,$$

for $g \in G$ and $a \in A$, on the simplicial level.

Then the actions of G and A are compatible, so that one has a (G, A) -baction, or equivalently, a $(G \times \bar{A})$ -action, where $\bar{A} = A^{\text{opposite}}$. The projection

$$\begin{array}{ccc} p: B(*, A, A) & \rightarrow & B(*, A, *) \\ \parallel & & \parallel \\ EA & & BA \cong EA/A \end{array}$$

is a $(G, A; \sigma)$ -covering, to be thought of as the universal $(G, A; \sigma)$ -principal covering.

In order to keep track of G and σ , we shall sometimes denote EA by $E_G^\sigma A$ and BA by $B_G^\sigma A$.

As examples, $B_G^1 G$ is BG acted upon by G via conjugation, while if G has order prime to p , then $B_G^\sigma Z_p$ must have σ trivial and hence a trivial G -action.

We would like to compute the Bredon equivariant cohomology of $B_G^\sigma A$, with coefficients in suitable Bredon coefficient systems. To do this, we shall need information about the orbit structure of the $(G \times \bar{A})$ -space $E_G^\sigma A$.

If $\psi: G \rightarrow A$ is a homomorphism and $H \subset G$, let $H_\psi \subset G \times \bar{A}$ be the subgroup $\{(h, \psi(h)^{-1})\}_{h \in H}$.

PROPOSITION 1.3. *Denote by ψ_a the homomorphism $G \rightarrow A$ given by $g \rightarrow a^{-1}\sigma(g)a$, for $a \in A$. Then any point x in $E_G^\sigma A$ has isotropy subgroup of the form H_{ψ_a} with $H \subset G$ the isotropy subgroup of $p(x)$.*

PROOF. Since the action on simplicial coordinates is trivial, it suffices to compute isotropy on the simplicial space level where it is obvious. \square

Returning to the case of the (G, A) -covering space $p: X \rightarrow X/A$, one sees again that for $x \in X$, $(G \times \bar{A})_x = \{(h, a_1^{-1}\sigma(h)a_1) : H \in G_{\{x\}}\}$, (where $\eta(a_1) = x$) $= (G_{\{x\}})_{\psi_{a_1}}$.

We shall concern ourselves with the computation of Bredon cohomology $H_G^*(X/A; T)$ for suitable coefficient systems T .

2. Hecke functors and equivariant cohomology. If \mathfrak{F} is a family of subgroups of G closed under conjugation, let $\mathbf{Z}(G, \mathfrak{F})$ denote the category whose objects are the G -sets G/H with $H \in \mathfrak{F}$ and whose morphisms are the $\mathbf{Z}G$ -module morphisms $\mathbf{Z}G/H \rightarrow \mathbf{Z}G/K$, where $\mathbf{Z}G/J$ denotes the free module on the G -set G/J .

Now let \mathfrak{F}' be a family, closed under conjugation, of subgroups of $G \times \bar{A}$ of the form H_ψ for $H \subset G$. Define an associated family \mathfrak{F} of subgroups of G by taking $\mathfrak{F} = \{\pi(Q) : Q \in \mathfrak{F}' \text{ and } \pi: G \times \bar{A} \rightarrow G \text{ the projection}\}$. As an example, one might take $\mathfrak{F}' = \{H_{\psi_a} : H \subset G \text{ and } a \in A\}$. If \mathfrak{F}' is closed under passage to subgroups, then so is \mathfrak{F} , and one has a projection, $P: \mathbf{Z}(G \times \bar{A}; \mathfrak{F}') \rightarrow \mathbf{Z}(G; \mathfrak{F})$ which assigns to $(G \times \bar{A})/Q$ the object $G/\pi(Q)$ and to $f: \mathbf{Z}(G \times \bar{A})/Q \rightarrow \mathbf{Z}(G \times \bar{A})/N$ the composite

$$\mathbf{Z}G/\pi(Q) \xrightarrow{u} \mathbf{Z}(G \times \bar{A})/Q \xrightarrow{f} \mathbf{Z}(G \times \bar{A})/N \xrightarrow{v} \mathbf{Z}G/\pi(N),$$

where $u(g\pi(Q)) = (g, \psi(g)^{-1})Q$ for any ψ such that Q has the form H_ψ , and where v is given by $(g, a)N \rightarrow g\pi(N)$. That this composite is independent of the choice of ψ and that P respects composition is easy to check.

As in [W1], there are associated notions of Hecke functors.

DEFINITION 2.1. $A(G, \mathfrak{F})$ -Hecke functor is an additive contravariant functor $T: \mathbf{Z}(G, \mathfrak{F}) \rightarrow \mathcal{A}b$, the category of abelian groups. Similarly, one has the notion of a $\mathbf{Z}(G \times \bar{A}; \mathfrak{F}')$ Hecke functor.

The collection of (G, \mathfrak{F}) -functors forms a category $\mathfrak{N}(G, \mathfrak{F})$, where the morphisms are taken to be the natural transformations.

Notice that P induces a functor $P^*: \mathfrak{N}(G; \mathfrak{F}) \rightarrow \mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$ in a natural way.

EXAMPLES 2.2. (i) If \hat{R} is a $\mathbf{Z}G$ -module, one has an associated (G, \mathfrak{F}) -Hecke functor \hat{R} given by $\hat{R}(G/H) = \hat{R}^H \cong \text{Hom}_{\mathbf{Z}G}(\mathbf{Z}G/H, R)$ for $H \in \mathfrak{F}$. If $f: \mathbf{Z}G/K \rightarrow \mathbf{Z}G/H$ is in $\mathbf{Z}(G; \mathfrak{F})$, then one defines $\hat{R}(f)$ to be composition with f . In particular, when $R = \mathbf{Z}$ with the trivial G -action, then $\hat{\mathbf{Z}}(G/H) = \mathbf{Z}$ for each $H \in \mathfrak{F}$.

(ii) If R is a $\mathbf{Z}(G \times \bar{A})$ -module, then the construction in (i), with G replaced by $G \times \bar{A}$, gives an associated $(G \times \bar{A}, \mathfrak{F}')$ -Hecke functor.

Now let M_* be a d.g. $(G \times \bar{A}, \mathfrak{F}')$ -Hecke functor, and define its cohomology, $H^*(M_*; T)$, with coefficients in $T \in \mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$, to be the cohomology of the Hom sets $\mathfrak{N}(G \times \bar{A}; \mathfrak{F}')(\mathfrak{N}_*, T)$.

EXAMPLES 2.3. (i) If X is a $(G \times \bar{A})$ -CW complex with orbit-types $(G \times \bar{A})/Q$ for $Q \in \mathfrak{F}'$, then the chain complex $C_*(X)$ of X is a $\mathbf{Z}(G \times \bar{A})$ -module, and hence, by Example 2.2(ii), a $(G \times \bar{A}, \mathfrak{F}')$ -Hecke functor. Define $\mathfrak{N}H_{G \times \bar{A}}^*(X; T)_{\mathfrak{F}'}$ to be $H^*(C_*(X); T)$ for $T \in \mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$. If \mathfrak{F}' is closed under passage to subgroups, we shall see that this coincides with Bredon cohomology, provided T , regarded as a $(G \times \bar{A})$ -equivariant coefficient system (see [B1]), extends to one defined on $(G \times \bar{A})/K$ for all $K \subset G \times \bar{A}$. More importantly, we shall see in §3 that

$$\mathfrak{N}H_{(G \times \bar{A})}^*(X; P^*T) \cong H_G^*(X/A; T)$$

for a Hecke functor T .

(ii) Let $T \in \mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$ be such that each $T(G/H)$ is free. One then has a projective resolution of T in $\mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$ given explicitly by the two-sided algebraic Bar construction in [W1] with G replaced by $G \times \bar{A}$. If we denote this resolution by $W_*(T, \mathfrak{F}')$, then set $H_{(G \times \bar{A})}^*(T; T')_{\mathfrak{F}'} = H^*(W_*(T, \mathfrak{F}'); T')$ for $T' \in \mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$ is clearly the derived functor Ext^* in the category $\mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$ and generalized the notion of the cohomology of G . Indeed, if $A = \{e\}$ and \mathfrak{F} is the collection of all subgroups of $G \times \bar{A} \cong G$, then $H_{(G \times \bar{A})}^*(\hat{\mathbf{Z}}; \hat{R})_{\mathfrak{F}} \cong H^*(G; R)$.

REMARK 2.4. The requirement that T be a Hecke functor may be dropped. In [W4], we deal with arbitrary coefficient systems T , and this involves a formal change of category from $\mathbf{Z}(G; \mathfrak{F})$. The restriction here to Hecke functors demonstrates the direct generalization of $H^*(G; R)$.

3. Relationship with Bredon cohomology. As promised in §2, we consider the theory $H_{(G \times \bar{A})}^*(X; T)_{\mathfrak{F}'}$ for \mathfrak{F}' as in §2.

By [W1, 3.1] one has

PROPOSITION 3.1. *Let X be any $(G \times \bar{A})$ -CW complex with cells of type $(G \times \bar{A})/Q$ with $Q \in \mathfrak{F}'$. Then $\mathfrak{N}H_{(G \times \bar{A})}^*(X; T)_{\mathfrak{F}'} \cong H_{(G \times \bar{A})}^*(X; T)$, Bredon cohomology of X with coefficients in the coefficient system T , where we take $T((G \times \bar{A})/Q) = 0$ if $Q \notin \mathfrak{F}'$.*

COROLLARY 3.2. *Let X be as in 3.1. Then $\mathfrak{N}H_{(G \times \bar{A})}^*(X; P^*T)_{\mathfrak{F}'} \cong H_G^*(X/A; T)$, where we regard T as the Bredon coefficient system as in Example 2.3(i).*

PROOF. By the proposition, we already have

$$\mathfrak{N}H_{(G \times \bar{A})}^*(X; P^*T)_{\mathfrak{F}'} \cong H_{(G \times \bar{A})}^*(X; P^*T)$$

and we are left with having to show that $H_{(G \times \bar{A})}^*(X; P^*T) \cong H_G^*(X/A; T)$, and this is standard. Let $\pi: G \times \bar{A} \rightarrow G$ be projection. Then $p: X \rightarrow X/A$ is a π -map, and, we claim, gives rise to an induced homomorphism in equivariant cohomology. Indeed, the identity maps $1: T(G/\pi(Q)) \rightarrow P^*T((G \times \bar{A})/Q)$ form a natural transformation with respect to π (see [II, p. 45]), and so, by Corollary II, 1.5 in [II], we have induced homomorphisms

$$(P, \pi, 1)^*: H_G^n(X/A; T) \rightarrow H_{(G \times \bar{A})}^n(X; P^*T)$$

for all n . When $X = (G \times \bar{A})/Q$, the projection $p: X \rightarrow X/A$ is the natural map $p_\pi: (G \times \bar{A})/Q \rightarrow G/\pi(Q)$, and by [II, II, 1.6], $(p, \pi, 1)^*$ corresponds for such an X , to the identity map $1: T(G/\pi(Q)) \rightarrow P^*T((G \times \bar{A})/Q) \stackrel{\text{def}}{=} T(G/\pi(Q))$ (under the dimension axiom isomorphism). It now follows that $(p, \pi, 1)^*$ is an isomorphism for any $(G \times \bar{A})$ -CW complex X . \square

COROLLARY 3.3. *Let X be as in Proposition 3.1, with X/Q \mathbf{Z} -acyclic for each $H \in \mathfrak{F}'$. Then*

$$H_G^*(X/A; T) \cong H_{(G \times \bar{A})}^*(\hat{\mathbf{Z}}; P^*T)_{\mathfrak{F}'},$$

provided \mathfrak{F}' is closed under passage to subgroups. Here, $\hat{\mathbf{Z}}$ denotes the constant functor in $\mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$.

PROOF. Since $C_*(X/H_\psi) \cong C_*(X)^{H_\psi}$ is a resolution of \mathbf{Z} for each $H \in \mathfrak{F}'$ it follows that, in $\mathfrak{N}(G \times \bar{A}; \mathfrak{F}')$, $C_*(X)$ is a projective resolution of \mathbf{Z} . Indeed, one notes that the assignment $C_*(X/-) \rightarrow H_0(X/-; \mathbf{Z})$ is in $\mathfrak{N}(G \times \bar{A}; \mathfrak{F}_\psi)$, the latter functor belonging there by virtue of the isomorphisms $H_0(X/H_\psi; \mathbf{Z}) = H_0(C^*(X)^{H_\psi}; \mathbf{Z})$. The result now follows by the definition in Example 2.3 (ii). \square

We may now conclude the main result. Let $\sigma: G \rightarrow A$ be a homomorphism, take ψ to be the collection $\{\psi_a: a \in A\}$ associated with σ , as in Proposition 1.3, and take \mathfrak{F} to be the collection of all subgroups of G , and $\mathfrak{F}' = \mathfrak{F}_\psi$.

THEOREM 3.4. *With the above conventions,*

$$H_G^*(B_G^\sigma A; T) \cong H_{(G \times \bar{A})}^*(\hat{\mathbf{Z}}; P^*T)_{\mathfrak{F}_\psi}.$$

PROOF. By Corollary 3.3, it suffices to show that $E_G A/H_{\psi_a}$ is contractible for each $H \subset G$ and $a \in A$. But $(E_G^\sigma A)^{H_{\psi_a}} = B(*, A, A)^{H_{\psi_a}} = B(*, C_A(H), C_A(H)a)$, where $C_A(H) = \{a \in A: \sigma(h)a\sigma(h)^{-1} = a \text{ for each } h \in H\}$, and $C_A(H)a$ denotes the right coset of a in $A/C_A(H)$. Since $B(*, C_A(H), C_A(H)a)$ is homeomorphic to the contractible space $EC_A(H)$, it follows that $(E_G^\sigma A)^{H_{\psi_a}}$ is contractible for each H and a , whence, by the H_{ψ_a} -equivariant version of Whitehead's theorem, $E_G^\sigma A$ is H_{ψ_a} -equivariantly contractible, so that $E_G A/H_{\psi_a}$ is contractible. \square

REMARKS 3.5. (i) If \mathfrak{F} is not closed into passage to subgroups, the argument in the proof of 3.4 fails as the Whitehead theorem does not apply.

(ii) If $G = \{1\}$, then the theorem implies the well-known result $H^*(BA; R) \cong H^*(A; R)$ for any trivial A -module R .

(iii) The theorem implies that $H_G^*(B_G^\sigma A; T)$ is computable from a purely algebraic viewpoint via the use of a two-sided Bar construction.

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