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MINIMAL TOPOLOGIES OF PARA-H-CLOSED SPACES[1]

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ABSTRACT. A Hausdorff space is para-H-closed if every open cover has a locally-finite open refinement (not necessarily covering the space) whose union is dense in the space. We prove that minimal locally-H-closed, minimal locally-para-H-closed and minimal para-H-closed spaces are all minimal-Hausdorff. We also show that para-H-closed-closed spaces are H-closed.

Introduction. All spaces considered in this paper are assumed to be Hausdorff.

DEFINITION 1. A Hausdorff space is *para-H-closed* if every open cover has a locally-finite open refinement (not necessarily covering the space) whose union is dense in the space.

DEFINITION 2. A space is *locally para-H-closed* if every point has a neighbourhood whose closure is para-H-closed.

DEFINITION 3. Let P be a property of topological spaces. A space is *minimal-P* if it has P and there is no coarser topology on the space having P.

DEFINITION 4. A space X is *feebly compact* if every locally-finite collection of open subsets of X is finite.

THEOREM A. *A space is feebly compact if and only if every countable filter-base of open subsets has an adherent point.*

Theorem A is a well-known result.

Para-H-closed spaces and locally para-H-closed spaces were defined and studied by the author in [**3**]. It was shown by M. P. Berri [**1**] that every minimal locally compact space is compact. C. T. Scarborough and R. M. Stephenson [**2**] proved that every minimal paracompact space is compact. In this paper we improve these results and give some further analysis of minimal-Hausdorff and H-closed spaces.

Main results.

THEOREM 1. *A space is minimal-Hausdorff if and only if it is minimal-locally-H-closed.*

PROOF. Let (X, τ) be a minimal-locally-H-closed space. It suffices to show that X is H-closed. Let Γ be the open filter generated by

$$\{\text{int}(\text{cl}(U)) \colon U \in \tau \text{ and } X \setminus \text{int}(\text{cl}(U)) \text{ is } H\text{-closed}\}.$$

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[1]The results of this paper are a part of Chapter 3 of the author's doctoral dissertation written under the supervision of Dr. R. W. Heath.

Choose a point q in X and fix it. Define a new topology τ' on X by the following neighbourhood systems. Let

$$\tau'(q) = \{O \cup F: q \in O \in \tau, F \in \Gamma\},$$

and let

$$\tau'(x) = \tau(x) = \{O \backslash \{q\} \in \tau: x \in O\}, \quad \text{for each } x \neq q.$$

Then (X, τ') is H-closed. But $\tau' \subset \tau$. Since τ is minimal-locally-H-closed, $\tau' = \tau$. Therefore X is H-closed.

THEOREM 2. *Every minimal-locally-para-H-closed space is minimal-Hausdorff.*

As a corollary to Theorem 2, we get the following theorem.

THEOREM 3. *Every regular, minimal-locally-para-H-closed space is compact.*

The proof of Theorem 2 is by the help of the following propositions.

PROPOSITION 1. *Every minimal-locally-para-H-closed space is para-H-closed.*

PROOF. Let (X, τ) be a minimal-locally-para-H-closed space. Choose q in X and fix it. Let Γ be the open filter generated by the set

$$\{\text{int}(\text{cl}(U)): U \in \tau \text{ and } X \backslash \text{int}(\text{cl}(U)) \text{ is para-}H\text{-closed}\}.$$

Let

$$\tau'(q) = \{O \cup K: q \in O \in \tau, K \in \Gamma\},$$

and let

$$\tau'(x) = \tau(x) = \{O \in \tau: x \in O\}, \quad \text{for each } x \neq q.$$

Let τ' be the topology generated by these neighbourhood systems. We show that τ' is para-H-closed. Let γ be a τ'-open cover of X. Let $O_q \in \tau(q)$, $K \in \Gamma$ such that $O_q \cup K \subset U$ for some $U \in \gamma$. Also $X \backslash K$ is para-H-closed. $\text{int}(\text{cl}(K)) = K$. Since $X \backslash \text{int}[\text{cl}(O_q \cup K)] \subset X \backslash K$ and is a closed domain, it is para-H-closed. Let

$$\gamma' = \{O \cap X \backslash \text{int}[\text{cl}(O_q \cup K)]: O \in \gamma\}.$$

There is a locally-finite open refinement λ of γ' in the subspace $X \backslash \text{int}[\text{cl}(O_q \cup K)]$, whose union is dense in it. Let

$$\xi = \{V \cap (X \backslash \text{cl}(O_q \cup K)): V \in \lambda\} \cup \{O_q \cup K\}.$$

Then ξ is a locally-finite open refinement of γ in τ', whose union is dense in X. Therefore (X, τ') is para-H-closed. This shows that (X, τ) is para-H-closed.

PROPOSITION 2. *Every minimal-locally-para-H-closed space is H-closed.*

PROOF. Let (X, τ) be a minimal-locally-para-H-closed space. Suppose X is not H-closed. Then there exists an open filter-base Γ with no adherent point.

Choose q in X. As before, define

$$\tau'(q) = \{O \cup F: q \in O \in \tau, F \in \Gamma\},$$

$$\tau'(x) = \tau(x) = \{O \in \tau: x \in O\}, \quad \text{for each } x \neq q.$$

Let τ' be the topology generated by these neighbourhood systems. Then $\tau' \subset \tau$, but $\tau' \neq \tau$. There is an $F_q \in \Gamma$ such that $q \notin \mathrm{cl}(F_q)$. Let $O_q \in \tau(q)$ such that $O_q \cap F_q = \phi$. Then $O_q \notin \tau'$.

We show that (X, τ') is locally para-H-closed. By Proposition 1, (X, τ) is para-H-closed. Therefore $\mathrm{cl}_{\tau'}(Q)$ is para-H-closed, for each $Q \in \tau'(q)$. Also for each $x \neq q$, there is $U_x \in \tau'(x)$ such that $q \notin \mathrm{cl}_{\tau'}(U_x)$. So $\mathrm{cl}_{\tau'}(U_x) = \mathrm{cl}_\tau(U_x)$ which is para-H-closed. This implies that (X, τ') is locally para-H-closed, which is a contradiction to the minimality of (X, τ). Therefore Γ has an adherent point. Therefore X is H-closed.

As a generalization of the theorem **[2]** that every minimal-paracompact space is compact, we prove below that every minimal-para-H-closed space is minimal-Hausdorff.

THEOREM 4. *A Hausdorff space is minimal-Hausdorff if and only if it is minimal-para-H-closed.*

PROOF. Let (X, τ) be a minimal-para-H-closed space. Since every feebly compact, para-H-closed space is H-closed, it suffices to show that (X, τ) is feebly compact.

Suppose not. Then there is an open filter-base $\Gamma = \{F_n\colon n \in \omega\}$ with no adherent point. Fix q in X. Define new neighbourhood systems as follows:

$$\tau'(q) = \{O \cup F_n\colon q \in O \in \tau, n \in \omega\},$$

and let

$$\tau'(x) = \tau(x) = \{O \backslash \{q\} : x \in O \in \tau\}, \quad \text{for each } x \neq q.$$

Then τ' is a Hausdorff topology on X and it is strictly coarser than τ.

We claim that τ' is para-H-closed. Let γ be a τ'-open cover X. Let λ be a locally-finite open refinement of γ in τ, whose union is dense in (X, τ). Since γ is a τ'-open cover of X, there exists $O_0 \in \tau(q)$ and an $n \in \omega$, such that $O_0 \cup F_n \subset U_0$, for some $U_0 \in \gamma$. Define $\xi = \{V \backslash \mathrm{cl}[F_n \cup O_0]\colon V \in \lambda\} \cup \{F_n \cup O_0\}$.

CLAIM (i). ξ is locally-finite open in (X, τ').

PROOF. Let $x \in X$. Since λ is locally-finite open in (X, τ), there is O_x open neighbourhood of x in τ such that O_x hits at most finitely many elements of λ. If $x = q$, then $O_0 \cup F_n \in \tau'(q)$ and ξ is locally-finite at q with respect to τ'. If $x \neq q$, then $O_x \in \tau'(x)$ itself illustrates local-finiteness of ξ with respect to τ'.

CLAIM (ii). $\bigcup \xi$ is dense in (X, τ').

PROOF. $\mathrm{cl}_{\tau'}(\bigcup \xi) = \mathrm{cl}_{\tau'}[(\bigcup \lambda) \backslash \overline{F_n \cup O_0}] \cup \mathrm{cl}_{\tau'}(F_n \cup O_0) = \mathrm{cl}_\tau(\bigcup \lambda) = X$. Therefore ξ is the required refinement of γ. (X, τ') is thus para-H-closed, which contradicts the minimality of τ. Hence (X, τ) must be feebly compact.

DEFINITION. A Hausdorff space is called *pHc-closed* if it is para-H-closed and is closed in any para-H-closed space containing it.

It is well known that every paracompact-closed space is compact **[2]**. The following theorem generalizes this result.

THEOREM 5. *A Hausdorff space is H-closed if and only if it is pHc-closed.*

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