MINIMAL TOPOLOGIES OF PARA-\(H\)-CLOSED SPACES\(^1\)

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ABSTRACT. A Hausdorff space is para-\(H\)-closed if every open cover has a locally-finite open refinement (not necessarily covering the space) whose union is dense in the space. We prove that minimal locally-\(H\)-closed, minimal locally-para-\(H\)-closed and minimal para-\(H\)-closed spaces are all minimal-Hausdorff. We also show that para-\(H\)-closed-closed spaces are \(H\)-closed.

Introduction. All spaces considered in this paper are assumed to be Hausdorff.

Definition 1. A Hausdorff space is para-\(H\)-closed if every open cover has a locally-finite open refinement (not necessarily covering the space) whose union is dense in the space.

Definition 2. A space is locally para-\(H\)-closed if every point has a neighbourhood whose closure is para-\(H\)-closed.

Definition 3. Let \(P\) be a property of topological spaces. A space is minimal-\(P\) if it has \(P\) and there is no coarser topology on the space having \(P\).

Definition 4. A space \(X\) is feebly compact if every locally-finite collection of open subsets of \(X\) is finite.

Theorem A. A space is feebly compact if and only if every countable filter-base of open subsets has an adherent point.

Theorem A is a well-known result.

Para-\(H\)-closed spaces and locally para-\(H\)-closed spaces were defined and studied by the author in [3]. It was shown by M. P. Berri [1] that every minimal locally compact space is compact. C. T. Scarborough and R. M. Stephenson [2] proved that every minimal paracompact space is compact. In this paper we improve these results and give some further analysis of minimal-Hausdorff and \(H\)-closed spaces.

Main results.

Theorem 1. A space is minimal-Hausdorff if and only if it is minimal-locally-\(H\)-closed.

Proof. Let \((X, \tau)\) be a minimal-locally-\(H\)-closed space. It suffices to show that \(X\) is \(H\)-closed. Let \(\tau\)' be the open filter generated by

\[
\{ \text{int}(\text{cl}(U)) : U \in \tau \text{ and } X \setminus \text{int}(\text{cl}(U)) \text{ is } H\text{-closed} \}.
\]

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\(^1\)The results of this paper are a part of Chapter 3 of the author's doctoral dissertation written under the supervision of Dr. R. W. Heath.
Choose a point \( q \) in \( X \) and fix it. Define a new topology \( \tau' \) on \( X \) by the following neighbourhood systems. Let
\[
\tau'(q) = \{ O \cup F: q \in O \in \tau, F \in \Gamma \},
\]
and let
\[
\tau'(x) = \tau(x) = \{ O \setminus \{ q \} \in \tau: x \in O \}, \text{ for each } x \neq q.
\]
Then \( (X, \tau') \) is \( H \)-closed. But \( \tau' \subset \tau \). Since \( \tau \) is minimal-locally-\( H \)-closed, \( \tau' = \tau \). Therefore \( X \) is \( H \)-closed.

**Theorem 2.** Every minimal-locally-para-\( H \)-closed space is minimal-Hausdorff.

As a corollary to Theorem 2, we get the following theorem.

**Theorem 3.** Every regular, minimal-locally-para-\( H \)-closed space is compact.

The proof of Theorem 2 is by the help of the following propositions.

**Proposition 1.** Every minimal-locally-para-\( H \)-closed space is para-\( H \)-closed.

**Proof.** Let \( (X, \tau) \) be a minimal-locally-para-\( H \)-closed space. Choose \( q \) in \( X \) and fix it. Let \( \Gamma \) be the open filter generated by the set
\[
\{ \text{int(cl(U))}: U \in \tau \text{ and } X \setminus \text{int(cl(U))} \text{ is para-}\( H \)-closed} \}.
\]
Let
\[
\tau'(q) = \{ O \cup K: q \in O \in \tau, K \in \Gamma \},
\]
and let
\[
\tau'(x) = \tau(x) = \{ O \in \tau: x \in O \}, \text{ for each } x \neq q.
\]
Let \( \tau' \) be the topology generated by these neighbourhood systems. We show that \( \tau' \) is para-\( H \)-closed. Let \( \gamma \) be a \( \tau' \)-open cover of \( X \). Let \( O_q \in \tau(q), K \in \Gamma \) such that \( O_q \cup K \subset U \) for some \( U \in \gamma \). Also \( X \setminus K \) is para-\( H \)-closed. \( \text{int(cl(K))} = K \). Since \( X \setminus \text{int(cl(O_q \cup K))} \subset X \setminus K \) and is a closed domain, it is para-\( H \)-closed. Let
\[
\gamma' = \{ O \cap X \setminus \text{int} [\text{cl}(O_q \cup K)]: O \in \gamma \}.
\]
There is a locally-finite open refinement \( \lambda \) of \( \gamma' \) in the subspace \( X \setminus \text{int(cl(O_q \cup K))} \), whose union is dense in it. Let
\[
\xi = \{ V \cap (X \setminus \text{cl}(O_q \cup K)): V \in \lambda \} \cup \{ O_q \cup K \}.
\]
Then \( \xi \) is a locally-finite open refinement of \( \gamma \) in \( \tau' \), whose union is dense in \( X \). Therefore \( (X, \tau') \) is para-\( H \)-closed. This shows that \( (X, \tau) \) is para-\( H \)-closed.

**Proposition 2.** Every minimal-locally-para-\( H \)-closed space is \( H \)-closed.

**Proof.** Let \( (X, \tau) \) be a minimal-locally-para-\( H \)-closed space. Suppose \( X \) is not \( H \)-closed. Then there exists an open filter-base \( \Gamma \) with no adherent point.

Choose \( q \) in \( X \). As before, define
\[
\tau'(q) = \{ O \cup F: q \in O \in \tau, F \in \Gamma \},
\]
\[
\tau'(x) = \tau(x) = \{ O \in \tau: x \in O \}, \text{ for each } x \neq q.
\]
Let \( \tau' \) be the topology generated by these neighbourhood systems. Then \( \tau' \subset \tau \), but \( \tau' \neq \tau \). There is an \( F_q \in \Gamma \) such that \( q \notin \text{cl}(F_q) \). Let \( O_q \in \tau(q) \) such that \( O_q \cap F_q = \emptyset \). Then \( O_q \notin \tau' \).

We show that \((X, \tau')\) is locally para-\(H\)-closed. By Proposition 1, \((X, \tau)\) is para-\(H\)-closed. Therefore \( \text{cl}_\tau(Q) \) is para-\(H\)-closed, for each \( Q \in \tau'(q) \). Also for each \( x \neq q \), there is \( U_x \in \tau'(x) \) such that \( q \notin \text{cl}_\tau(U_x) \). So \( \text{cl}_\tau(U_x) = \text{cl}_\tau(U_q) \) which is para-\(H\)-closed. This implies that \((X, \tau')\) is locally para-\(H\)-closed, which is a contradiction to the minimality of \((X, \tau)\). Therefore \( \Gamma \) has an adherent point. Therefore \( X \) is \( H\)-closed.

As a generalization of the theorem \([2]\) that every minimal-paracompact space is compact, we prove below that every minimal-para-\(H\)-closed space is minimal-Hausdorff.

**Theorem 4.** A Hausdorff space is minimal-Hausdorff if and only if it is minimal-para-\(H\)-closed.

**Proof.** Let \((X, \tau)\) be a minimal-para-\(H\)-closed space. Since every feeably compact, para-\(H\)-closed space is \( H\)-closed, it suffices to show that \((X, \tau)\) is feeably compact.

Suppose not. Then there is an open filter-base \( \Gamma = \{F_n: n \in \omega\} \) with no adherent point. Fix \( q \in X \). Define new neighbourhood systems as follows:

\[
\tau'(q) = \{O \cup F_n: q \in O \in \tau, n \in \omega\}
\]

and let

\[
\tau'(x) = \{O \in \tau : x \in O \in \tau\}, \quad \text{for each } x \neq q.
\]

Then \( \tau' \) is a Hausdorff topology on \( X \) and it is strictly coarser than \( \tau \).

We claim that \( \tau' \) is para-\(H\)-closed. Let \( \gamma \) be a \( \tau'\)-open cover \( X \). Let \( \lambda \) be a locally-finite open refinement of \( \gamma \) in \( \tau \), whose union is dense in \((X, \tau)\). Since \( \gamma \) is a \( \tau'\)-open cover of \( X \), there exists \( O_0 \in \tau(q) \) and an \( n \in \omega \), such that \( O_0 \cup F_n \subset U_0 \), for some \( U_0 \in \gamma \). Define \( \xi = \{V \setminus \text{cl}(F_n \cup O_0): V \in \lambda \} \cup \{F_n \cup O_0\} \).

**Claim (i).** \( \xi \) is locally-finite open in \((X, \tau')\).

**Proof.** Let \( x \in X \). Since \( \lambda \) is locally-finite open in \((X, \tau)\), there is \( O_x \) open neighbourhood of \( x \) in \( \tau \) such that \( O_x \) hits at most finitely many elements of \( \lambda \). If \( x = q \), then \( O_0 \cup F_n \in \tau'(q) \) and \( \xi \) is locally-finite at \( q \) with respect to \( \tau' \). If \( x \neq q \), then \( O_x \in \tau'(x) \) itself illustrates local-finiteness of \( \xi \) with respect to \( \tau' \).

**Claim (ii).** \( \cup \xi \) is dense in \((X, \tau')\).

**Proof.** \( \text{cl}_{\tau'}((\cup \lambda) \setminus (F_n \cup O_0)) \cup \text{cl}_{\tau}((F_n \cup O_0)) = \text{cl}_{\tau}(\cup \lambda) = X \). Therefore \( \xi \) is the required refinement of \( \gamma \). \((X, \tau')\) is thus para-\(H\)-closed, which contradicts the minimality of \( \tau \). Hence \((X, \tau)\) must be feeably compact.

**Definition.** A Hausdorff space is called **\( pHc \)-closed** if it is para-\(H\)-closed and is closed in any para-\(H\)-closed space containing it.

It is well known that every paracompact-closed space is compact \([2]\). The following theorem generalizes this result.
Theorem 5. A Hausdorff space is $H$-closed if and only if it is $pHc$-closed.

Bibliography


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