AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 2, June 1983

REMARK ON CONTINUOUS COLLECTIONS

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ABSTRACT. The result generalizes an earlier theorem of the author which showed that pointwise periodic homeomorphisms of a connected manifold are periodic.

Newman's work on orbits of periodic transformations [**4**] has been extended to continuous collections of finite sets by Černavskii [**1**], (see also [**2**]). The extension does not appear to apply directly to orbits of pointwise periodic transformations [**3**]. It is shown here that the assumption of continuity in [**1**] can be replaced by a slightly weaker assumption which implies continuity, and then orbits of pointwise periodic transformations become included. These results are for manifolds.

Denote by H a collection of nonoverlapping finite sets filling a connected manifold M. All considerations here reduce to the case where M has no boundary and this is assumed. Let $H(x)$ be the finite set of the collection which includes x, and let $H(U)$ be the union of all $H(x)$, $x \in U$. If $H(U) = U$, U is called saturated. The number of points in $H(x)$ is called the multiplicity of x or $H(x)$ and is denoted by $\alpha(x)$.

If H is continuous, [**1**] shows it has bounded multiplicity and also that points of maximum multiplicity are everywhere dense. These facts are used here to give the same conclusion with a slightly weaker hypothesis. Using [**1**] the argument is analogous to [**3**] where the basis is [**4**].

THEOREM. *Let H be a collection of nonoverlapping finite sets filling a connected manifold M. Then H is continuous and, hence, of bounded multiplicity if it satisfies*

(a) *H is lower semicontinuous,*

(b) *H is continuous on a satuated subset if $\alpha(x)$ is bounded on the subset.*

PROOF. The function $\alpha(x)$ is lower semicontinuous by (a) and so its points of continuity are everywhere dense. They form an open set since α has integer values. Let K be the closed set where the least upper bound of α is infinite. Then α is locally bounded on $M - K$, and, by (b), H is continuous on $M - K$.

There is a decomposition space X^* of $M - K$ and a proper, open, and closed map $f\colon M - K \to X^*$. Notice that X^* is locally connected because $M - K$ is locally connected. The following lemma is known and can also be verified by the reader.

Received by the editors February 22, 1982.
1980 *Mathematics Subject Classification*. Primary 57S99, 57S19.
Key words and phrases. Finite-to-one mappings, continuous collections, pointwise periodic homeomorphisms.

LEMMA. *Let U^* be an open connected set in X^*. Then $f^{-1}(U^*)$ is open and each component maps onto U^*, that is, each component of $f^{-1}(U^*)$ intersects $f^{-1}(x^*)$ for all $x^* \in U^*$. The number of such components is finite.*

To check the lemma, observe that if V is a component of $f^{-1}(U^*)$, then $f(V) = V^*$ is an open connected set in U^*. If $V^* \neq U^*$, let x^* be a point of U^* in the boundary of V^*. Choose $x_i^* \in V^*$ where $x_i^* \to x^*$. Then choose $x_i \in f_{-1}(x_i^*)$ so that $x_i \to x$, $x_i \in V$. But then $x \in f^{-1}(U^*)$ and, hence, $x \in V$ because $V \cup \{x\}$ is connected. The rest of the lemma follows.

To proceed with the proof of the Theorem, let U be a component of $M - K$. Define a collection H' in U by $H'(x) = H(x) \cap U$, for $x \in U$. Then H' is continuous on U by (b). Since U is a manifold, it follows from Theorem 1 of [**1**] that H' is bounded on U. A similar fact is true for each component of $H(U)$. Because $H(U)$ has only a finite number of components, H is bounded on $H(U)$.

The set K is nowhere dense. The function $\alpha \mid K$, like α, is locally constant (on K) at a point of continuity. If U is the only component of $M - K$, then $H'(x) = H(x)$ and both are bounded in U. If K is nonnull, let p be a point of K where $\alpha \mid K$ is continuous. Then $\alpha(x)$ is bounded in a neighborhood of p by the bound of $\alpha(x)$ in U. Hence K is null, $U = M$, and the Theorem is true.

Assume now the Theorem is false so that K must separate M. Let p be a point of K where K separates M locally and where $(\alpha \mid K)(x)$ is continuous.

Let $p_1 = p, p_2, \ldots, p_{\alpha(p)}$ be the points of $H(p)$. At each p_i, K separates locally because K has locally homeomorphic neighborhoods (in K) at these points. Let V_i, $i = 1, \ldots, \alpha(p)$, be open connected sets in M satisfying:

(1) $p_i \in V_i$;
(2) $V_i - K$ is not connected;
(3) $(\alpha \mid K)(x) = \alpha(p)$, $x \in K \cap V_i$;
(4) for $x \in K \cap V_i$, $H(x) \cap V_i = \{x\}$;
(5) $\overline{V}_i \cap \overline{V}_j = \varnothing$, $i \neq j$.

Let U be a component of $M - K$ which intersects V_1. An argument will now be made concerning this component U which will then apply to any component of $M - K$ which intersects V_1.

Now $\overline{U} \cap (V_1 \cap K) \neq \varnothing$. Let q be a point of $\overline{U} \cap (V_1 \cap K)$ where this set separates locally. The collection H is bounded on

$$Z = H\left[U \bigcup_{i=1}^{\alpha(p)} (K \cap V_i)\right]$$

and, by (b), is continuous on Z. There is a continuous open map f from Z to its decomposition space Z^*—$f\colon Z \to Z^* = f(Z)$. Choose a neighborhood of $f(q) = q^*$ in the space Z^* and let Y^* be this neighborhood with $\cup_{i=1}^{\alpha(p)} f(K \cap V_i)$ omitted. Assume the choice made so that $f^{-1}(Y^*) \subset \cup V_i$. Choose a connected open neighborhood V of q with $V \subset V_1$ and (closure V) $\cap H(U) \subset f^{-1}(Y^*)$. If closure $H(U)$ included q as an inner point, then $\alpha(x)$ would be bounded at q, and q could not be in K by definition. Therefore $V -$ closure $H(U)$ contains an inner point.

Define W as follows:

$$W = \cup \{\text{all components of } f^{-1}(Y^*) \text{ which intersect } V\}.$$

Then $W \cup V$ is an open connected set in V_1. On $W \cup V$ define a collection H'' as follows:

$$\text{for } x \in W, H''(x) = H(x) \cap W;$$

$$\text{for } x \in V - W, \qquad H''(x) = \{x\}.$$

LEMMA. *The collection H'' is continuous.*

Let w be a point of W. Then $H''(w) = H(w) \cap W$. If y is near w, $H(y)$ is near $H(w)$, and then $H(y) \cap W$ is near $H(w) \cap W$ because W is open. Next suppose $v \in V - W$. Then $H''(v) = \{v\}$. If v is an inner point of $V - W$, H'' is trivial at all points near v so H'' is continuous at v.

Next suppose $v \in V - W$ and v is not an inner point of $V - W$; then H'' is trivial at all points of $V - W$ and so continuous at v on $V - W$. Now let w_n be a sequence of points in W, $\lim w_n = v$. The point v is in $K \cap V_1$ so $H(w_n) \to H(v)$. Now v is the only point of $H(v)$ in V_1. Hence, $H(w_n) \cap V_1 \to \{v\}$ and then $H''(w_n) \to \{v\}$. This proves the Lemma.

Since H'' is trivial on $V - W$, which contains an inner point, we see that H'' is trivial everywhere on $W \cup V$. It follows that $x \in V \cap U$ implies $H(x) \cap W = \{x\}$, that is, $x \in V \cap U$ implies $H(x) \cap V = \{x\}$.

Let $x \in V \cap U$, and let I be an arc in V joining x to a point q^1 in $K \cap V$ and which lies in U except for q^1. The collection H is continuous on $H(I)$. If x_i is any point of $H(x)$ in V_i, there is an arc I_i joining x_i to q_i^1, where $q_i^1 \in H(q)$ and where I_i lies in $H(I) \subset f^{-1}(Y_i^*) \cup H(q)$. The existence of such an arc follows from p. 231 of Montgomery and Zippin, *Topological transformation groups*.

Next choose a connected open neighborhood of q_i^1, say V_i^1, with (closure V_i^1) $\cap$ $H(U) \subset f^{-1}(Y^*)$, and let

$$W_i = V_i^1 \cup \{\text{all components of } f^{-1}(Y^*) \text{ which intersect } V_i^1\}.$$

As before, it follows that x_i is the only point of $H(x)$ in W_i. Thus each $q_i^1 \in H(q^1)$ has a neighborhood which contains at most one point of $H(x)$. There are $\alpha(q^1)$ such neighborhoods and their union contains $H(x)$. Hence $\alpha(x) \leqslant \alpha(q^1) = \alpha(q) = \alpha(p)$. This is true for every component which touches V_1, so $\alpha(x)$ is bounded at p and p is not in K. This contradiction shows that K does not separate and, in fact, is null. Then H is bounded as the Theorem states.

The author has benefited from helpful comments by L. McAuley and E. Robinson.

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