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MULTIPLIER REPRESENTATIONS OF DISCRETE GROUPS

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ABSTRACT. Let σ be a multiplier on the discrete group G. Extending theorems of Kaniuth and Thoma to the case of multiplier representations, we determine when the left regular σ representation of G has a type I subrepresentation, and when all the σ representations of G are type I.

1. Let σ be a normalized multiplier on the discrete group G (cf. [**1, 7, 9**]). An element $x \in G$ is *σ-regular* if $\sigma(x, a) = \sigma(a, x)$ for all a which commute with x. If x is σ-regular so is every conjugate of x [**7**, Lemma 3], and we may speak of the σ-regular conjugacy classes in G. Let $\Delta_0 = \Delta_0(\sigma)$ be the set of elements lying in finite σ-regular conjugacy classes, let $\Delta = \Delta(\sigma)$ be the subgroup generated by Δ_0, and let Δ' be its commutator subgroup. The left regular σ representation $\lambda_\sigma = \lambda_{G,\sigma}$ acts on $l^2(G)$ by

$$\lambda_\sigma(x)f(y) = \sigma(x^{-1}, y)f(x^{-1}y), \qquad f \in l^2(G), x, y \in G.$$

It is a finite representation [**7**, Theorem 9]. We say that the multiplier σ is type I if all the primary σ representations are type I.

THEOREM 1. *λ_σ has a type* I *subrepresentation if and only if the following three conditions are satisfied*:

(a) $[G : \Delta] < \infty$.

(b) $|\Delta'| < \infty$.

(c) *σ is trivial on some cofinite subgroup.*

THEOREM 2. *The following are equivalent*:

(a) *λ_σ is type* I.

(b) *σ is type* I.

(c) *σ is trivial on some cofinite abelian subgroup.*

For the case of ordinary representations ($\sigma = 1$), Theorem 1 is due to Kaniuth [**6**] and Theorem 2 to Thoma [**15, 16**]. Smith [**13**] gave a global proof of these theorems, and the proof of the necessity of (a) and (b) in Theorem 1 is essentially her proof. Though it is quicker to use Thoma's theorem in the proof of Theorem 2, we indicate another method which gives an alternate proof of Thoma's theorem. Duflo [**3**] has also found a version of Theorem 2. After completing most of this work, I learned that A. K. Holzherr [**4**] had also found, by somewhat different methods, Theorem 2,

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most of Theorem 1 and a number of other related results. In particular, he identified the projection on the type I part of λ_σ. I am indebted to the referee for pointing out the correct formulation of Theorem 1.

2. For each subset $S \subset G$ we denote by $Z(S) = Z_G(S)$ its centralizer in G and we put

$$Z_\sigma(S) = \{y \in Z(S) \mid \sigma(x, y) = \sigma(y, x), \text{all } x \in S\}.$$

If $S = \{x\}$ we write shortly $Z(x)$ or $Z_\sigma(z)$. x is σ regular if and only if $Z_\sigma(x) = Z(x)$. Note that $x \in Z_\sigma \Leftrightarrow y \in Z_\sigma(x)$ and $Z_\sigma(S) = \bigcap_{x \in S} Z_\sigma(x)$.

LEMMA 1. (a) *For each subset $S \subset G$, $Z_\sigma(S)$ is a subgroup.*
(b) *If H is a finitely generated subgroup of Δ then $[G\colon Z_\sigma(H)] < \infty$.*

PROOF. Osima [**11**] observed that $y \to \sigma(x, y)\sigma(y, x)^{-1}$, $y \in Z(x)$, is a character of $Z(x)$. $Z_\sigma(x)$ is the kernel of this character. Thus $Z_\sigma(x)$ and $Z_\sigma(S)$ are subgroups. It is easy to check that $Z_\sigma(a) \cap Z_\sigma(b) \subset Z(ab)$. If $a \in \Delta_0$, $Z_\sigma(a) = Z(a)$ is cofinite. If $x = a_1 a_2 \cdots a_n \in \Delta$, then $\bigcap_1^n Z_\sigma(a_i) \subset Z_\sigma(x)$. Thus $Z_\sigma(x)$ is cofinite. If H is generated by $\{x_1, \ldots, x_m\} \subset \Delta$ and $y \in H$, then $\bigcap Z_\sigma(x_i) \subset Z_\sigma(y)$, and $\bigcap Z_\sigma(x_i) \subset \bigcap_{y \in H} Z_\sigma(y) = Z_\sigma(H)$ is also cofinite.

3. To prove that (a), (b), (c) in Theorem 1 are necessary, we begin by observing that if the set Δ_0 is a subgroup of infinite index, then by [**7**, Theorem 4] λ_σ is type II. However, the proof of that theorem is valid also in the case that the subgroup Δ generated by Δ_0 is of infinite index. Thus in what follows we shall assume Δ is cofinite. Let $\mathfrak{A}(G)$ be the bicommutant of $\lambda_\sigma(G)$, and if E is a projection in $\mathfrak{A}(G)'$, let $\mathfrak{A}_E(G)$ be the corresponding reduced algebra.

LEMMA 2 (SMITH [**13**]). *Let G be an FC group, H a subgroup, E a central projection in $\mathfrak{A}(G)$ so that $\mathfrak{A}_E(G)$ and $\mathfrak{A}_E(H)$ are type I_n, for some $n < \infty$. Then $[Z_\sigma(H), G]$ is finite.*

Smith's proof of this lemma [**13**, Lemma 2] for the case $\sigma = 1$ may be adopted, after replacing the group algebra by the appropriate twisted group algebra, almost without change to the present setting.

Observing that Δ is contained in the subgroup of elements lying in finite conjugacy classes, we see that the remainder of Smith's proof for the case $\sigma = 1$ [**13**, pp. 403–404] that if λ_σ has a type I summand, then $|\Delta'| < \infty$, is valid for arbitrary σ. This proves the necessity of (a) and (b) in Theorem 1.

To show there is a cofinite subgroup of G on which σ is trivial, it is enough to show there is a cofinite subgroup of Δ on which σ is trivial. Thus we may suppose $G = \Delta$ and $[G, G] = G'$ is finite. Put $H = Z_\sigma(G')$ and $F = G' \cap H$. By Lemma 1, H is cofinite. F is finite and $[H, H] \subset F \subset \text{center}(H)$.

LEMMA 3. *$\sigma|_{H \times H}$ is similar to a multiplier lifted from a multiplier σ' on H/F.*

PROOF. If γ is an irreducible σ representation of H, then for all $a \in F$ and $x \in H$,

$$\gamma(a)\gamma(x) = \sigma(a, x)\gamma(ax) = \sigma(x, a)\gamma(xa) = \gamma(x)\gamma(a).$$

Hence $\gamma(a)$ is a multiple $\varphi(a)$ of the identity, i.e. $\gamma(a) = \varphi(a) \cdot 1$, where $|\varphi(a)| = 1$. Thus $F \subset$ projective kernel of γ [**2**, §1], and $\sigma|_{H\times H}$ is similar to a multiplier lifted from a multiplier σ' on H/F [**2**, Lemma 1.3].

Replacing $\sigma|_{H\times H}$ by that multiplier, we may suppose $\sigma(x, y) = \sigma(\bar{x}, \bar{y})$, all $x, y \in H$, where $\bar{x} = xF$.

Because λ_σ is finite and has a type I part, it contains a finite-dimensional subrepresentation, all of whose matrix coefficients vanish at ∞. The restriction of this representation to H contains an irreducible subrepresentation π, all of whose matrix coefficients vanish at ∞. If $n = \dim \pi$, then $\sigma^n(x, y) = \det \pi(x) \det \pi(y) \det \pi(xy)^{-1}$. Thus we may replace σ by a similar multiplier and assume $\sigma^n = 1$; that is, we may assume σ takes its values in a finite subgroup C of the circle.

Consider now the extension $1 \to C \to H^\sigma \to H \to 1$ of C by H defined by σ. Put $\pi^0(t, x) = t\pi(x)$, $x \in H$, $t \in C$. Then π^0 is a representation of the discrete group H^σ, all of whose matrix coefficients vanish at ∞. By a theorem of Taylor [**14**, Theorem 1, Corollary 1], there is a finite normal subgroup K of H^σ such that H^σ/K is a Moore group (only finite-dimensional unitary representations) and $K \subset \ker \pi^0$, i.e., π^0 is lifted from a representation of H^σ/K. H^σ/K is, in particular, type I, and by Thoma's theorem [**13**, **15**, **16**] there is a cofinite subgroup $M \subset H^\sigma$ so that M/K is abelian. Then $P = M/K \cap (H/H \cap K)$ is a cofinite abelian subgroup of $H/H \cap K \subset H^\sigma/K$. Since $K \subset \ker \pi^0$, $K \cap H \subset \ker \pi \subset$ projective kernel of π. Thus $\sigma|_{H\times H}$ is similar to a multiplier lifted from a multiplier σ'' of $H/H \cap K$ [**2**, Lemma 1.3]. Again replacing σ by a similar multiplier, we may think of π as a σ'' representation of $H/H \cap K$. Then $\pi|_P$ is a finite-dimensional σ'' representation of the abelian group P. Hence there is a cofinite subgroup $Q \subset M$, $H \cap K \subset Q$, so that σ'' is trivial on $Q/H \cap K$ [**2**, Lemma 2.1, Theorem 3.1]. Then $\sigma|_{Q\times Q}$ is trivial. We have the inclusion $Q \subset M \subset H \subset G$, and since each subgroup is of finite index in the next, Q is cofinite in G.

This completes the proof of the necessity of the conditions in Theorem 1. To prove that they are sufficient we begin with

LEMMA 4. *Let H be a subgroup of finite index in G and π a finite type* I *σ-representation of H. Then σ-$\mathrm{Ind}_H^G \pi$ is type* I.

(For the definition of σ-induction see [**2** or **9**].)

PROOF. H contains a cofinite normal subgroup N and $\pi|_N$ is a finite type I representation. By the Mackey machine, σ-$\mathrm{Ind}_N^G \pi|_N$ is also a finite type I representation, and it is easy to see that σ-$\mathrm{Ind}_H^G \pi \subset \sigma$-$\mathrm{Ind}_N^G \pi|_N$.

Let Q be a cofinite subgroup of G on which σ is trivial. Then $\Delta \cap Q' = \Delta_Q$ is cofinite, $|\Delta'_Q| < \infty$, and because Δ is contained in the subgroup of G of elements in finite conjugacy classes, each $x \in \Delta_Q$ lies in a finite conjugacy class. By Kaniuth's theorem λ_{Δ_Q} (the left regular representation of Δ_Q) has a type I part. By Lemma 4, $\lambda_0 = \sigma$-$\mathrm{Ind}_{\Delta_Q}^G \lambda_\sigma$ has a type I part. This completes the proof of Theorem 1.

This theorem can be presented in slightly different terms. For any group H let H_{FC} be the subgroup of elements lying in a finite conjugacy class. Note that $\Delta \subset G_{FC}$.

THEOREM 1′. λ_σ *has a type* I *part if and only if the following three conditions are satisfied*:

(a′) $[G: G_{FC}] < \infty$.

(b′) $|G'_{FC}| < \infty$.

(c′) σ *is trivial on some cofinite subgroup.*

In fact, suppose λ_σ has type I part. Then $[G:\Delta] < \infty$ and because $\Delta \subset G_{FC}$, $[G: G_{FC}] < \infty$. Since $|\Delta'| < \infty$ and $[G_{FC}: \Delta] < \infty$, $|G'_{FC}| < \infty$ [**10**, Lemma 4.1], and by Theorem 1, (c) is satisfied. Conversely, if σ is trivial on a cofinite subgroup Q, then $Q_{FC} = Q \cap G_{FC}$ is cofinite in Q, and $Q'_{FC} \subset G'_{FC}$ is also finite. By Kaniuth's theorem [**6, 13**], λ_Q has a type I part, and by Lemma 4, λ_σ has a type I part.

4. The implication (c) ⇒ (b) of Theorem 2 is a consequence of the Mackey machine [**2, 9**]. In fact, if H is a cofinite normal subgroup for which $\sigma|_{H\times H}$ is type I, e.g., an abelian subgroup on which σ is trivial, it follows from the Mackey machine that σ is type I, and every cofinite subgroup contains a cofinite normal subgroup. (b) ⇒ (a) is clear (see [**12**] for the case of uncountable groups). Thus what remains to show is (a) ⇒ (c).

LEMMA 5. *If* $\lambda_{\sigma,G}$ *is type* I *so is* $\lambda_{\sigma,H}$ *for every subgroup* H.

This lemma is, in fact, valid for open subgroups of a locally compact group and is what is proved by Kallman in the proof of Proposition 2.4 of [**5**].

To complete the proof of Theorem 2 we need only observe that if Q is a cofinite subgroup of G on which σ is trivial, then $\lambda_{\sigma,Q} = \lambda_Q$ is type I by Lemma 5, and by Thoma's theorem [**13, 15, 16**] Q contains a cofinite abelian subgroup.

5. It is possible to show (a) ⇒ (c) in Theorem 2 without use of Thoma's theorem and obtain, in particular, another proof of that theorem. Assume λ_σ is type I. Then by Theorem 1, Δ is cofinite and Δ' is finite. (This part of Theorem 1 did not use Thoma's theorem.) It is enough to show Δ contains a cofinite abelian subgroup on which σ is trivial. Thus we may suppose that $G = \Delta$ and G' is finite. As in §2 we put $H = Z_\sigma(G')$, $F = H \cap G'$ and assume that $\sigma(x, y) = \sigma'(\bar{x}, \bar{y})$, $x, y \in H$, where $\bar{x} = xF$. Let $p: H \to H/F$ be the canonical map.

We are assuming that $\lambda_{\sigma,G}$ is type I. By Lemma 5, $\lambda_{\sigma,H}$ is type I. $\lambda_{\sigma',H/F} \circ p$ is a subrepresentation of $\lambda_{\sigma,H}$ (the map $f \to |F|^{-1} f \circ p$ is the intertwining operator). Thus $\lambda_{\sigma',H/F}$ is also type I. But H/F is abelian and we know [**2**, Theorem 3.1, Lemma 3.1] that $\lambda_{\sigma',H/F}$ is type I if and only if there is a cofinite subgroup $K \subset H$ so that $\sigma'|_{K/F,K/F}$ is trivial, or what is the same, $\sigma|_{K\times K}$ is trivial. Replacing σ by a similar multiplier, we may suppose $\sigma|_{K\times K} = 1$. Then $\lambda_{\sigma,K} = \lambda_K$ is also type I (Lemma 5).

We now have an extension $1 \to F \to K \to P \to 1$, where F is finite and central in K, P is abelian, and K has a type I regular representation. We apply the Mackey machine to describe λ_K. We have

$$\lambda_K = \operatorname{Ind}_F^K \lambda_F = \bigoplus_{\theta\in\hat{F}} \operatorname{Ind}_F^K \theta = \bigoplus_{\theta\in\hat{F}} \pi_\theta,$$

where $\pi_\theta = \operatorname{Ind}_F^K \theta$. Thus λ_K is type I if and only if each π_θ is type I. Since K centralizes F, each $\theta \in \hat{F}$ may be extended to a projective character θ' with Mackey obstruction ω_θ^{-1}, and $\pi_\theta = \theta' \otimes \rho_\theta''$, where ρ_θ is the right regular ω_θ representation of P (here we think of ω_θ as a multiplier on P), and ρ_θ'' is its lift to K (see [**1**]). Thus each π_θ is type I if and only if each ρ_θ is type I, and this happens if and only if ω_θ is trivial on a cofinite subgroup $Q_\theta \subset P$ [**2**, Theorem 3.1, Lemma 3.1]. Let C be the inverse image in K of $\bigcap_{\theta \in \hat{F}} Q_\theta$. C is cofinite. We shall show it is abelian. Let $\pi \in \hat{C}$. Since F is central in C, $\pi(a) = \theta_\pi(a) \cdot 1$, where $\theta_\pi \in \hat{F}$, $a \in F$. Then $x \to \theta_\pi'^{-1}(x)\pi(x)$ is constant on the F cosets in C and defines a representation π^* of $C/F \subset P$ with multiplier ω_θ. Since ω_θ is trivial on C/F, $\omega_\theta(x, y) = \omega_\theta(y, x)$, all $x, y \in C$ and $\pi^*(x)\pi^*(y) = \pi^*(y)\pi^*(x)$, all $x, y \in C/F$. Since π^* is also irreducible, $\dim \pi^* = 1 = \dim \pi$. Thus all irreducible representations of C are 1-dimensional and C is abelian.

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