PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 2, June 1983

SHORTER NOTES

The purpose of this department is to publish very short papers of unusually elegant and polished character, for which there is no other outlet.

SCHRÖDINGER OPERATOR METHODS IN THE STUDY OF A CERTAIN NONLINEAR P.D.E.

E. M. HARRELL[1] AND B. SIMON[2]

ABSTRACT. We prove that $\Delta u + hu^{\alpha} = 0$ has no positive solutions for certain h, α by studying the linearized equation $(\Delta + hu^{\alpha-1})\psi = e\psi$.

In this note we show that some of the work of Gidas and Spruck [**2**] on the absence of positive solutions of

$$\Delta u(x) + h(x)u^{\alpha}(x) = 0 \tag{1}$$

can be recovered with simple arguments about the number of eigenvalues of linear operators.

THEOREM. *Let D be the domain $\{|x| > r_0\}$ of $\mathbf{R}^n$, $n \geq 3$, and h a locally L^{∞} positive function satisfying $h(x) \geq \text{const}\,|x|^{\sigma}$, $\sigma > -2$. If $1 < \alpha < (n+\sigma)/(n-2)$, then no positive C^2 function u satisfies* (1).

PROOF. It has been shown by Allegretto [**1**] and Piepenbrink [**3**] (see also §C8 of [**6**]) that the existence of positive solutions of an equation $(-\Delta + q(x))u = 0$ (conventionally with a different sign from (1)) on an exterior domain implies that $\dim P_{(-\infty,0)}(-\Delta + q(x)) < \infty$, where $P_{(-\infty,0)}(-\Delta + q(x))$ is the spectral projection for the open interval $(-\infty, 0)$ associated with any selfadjoint realization of $-\Delta + q$. In other words, there are only a finite number of negative bound states. On the other hand, suppose that u is a positive solution of (1) on D. Then, since $\Delta u < 0$, the subharmonic comparison argument of [**4**] shows that $u(x) > cg_R(x)$ for some constant c and $r_0 < |x| \leq R$, where $g_R(x)$ is the Green function for $\Delta g_R = -\delta(x)$ on $\{|x| \leq R\}$ with Dirichlet B.C. at $|x| = R$. The constant c is such that $u(x) > cg_R(x)$ when $|x| = r_0$. Since $g_R(x)$ increases monotonically to $1/\omega_n |x|^{n-2}$ as $R \uparrow \infty$ on D,

Received by the editors July 14, 1982.
1980 *Mathematics Subject Classification*. Primary 35J60, 35J15.
[1]Research partially supported by U.S.N.S.F. grant MCS-79-26408.
[2]Research partially supported by U.S.N.S.F. grant MCS-81-19979.

0002-9939/82/0000-1152/$01.50

it follows that $u(x) \geqslant \text{const}\,|x|^{-(n-2)}$. Therefore

$$-h(x)u^{\alpha-1} < -\text{const}\,|x|^{\sigma-(\alpha-1)(n-2)} < -\text{const}\,|x|^{-2+\varepsilon},$$

where $\varepsilon > 0$. But any potential $q(x) < -\text{const}\,|x|^{-2+\varepsilon}$ gives rise to an *infinite* number of negative bound states by the min-max principle **[5]**, so there is a contradiction when we identify $q(x) = -hu^{\alpha-1}$. $\square$

REMARKS. 1. Δ may be replaced with a more general elliptic operator $\partial_i a_{ij}(x)\partial_j$ such that $a_{ij} \in C^2$ and positive definite for each x and satisfying a growth condition **[3]**.

2. h need only be locally L^p with $p > n/2$. Also one only needs a weak solution with $hu^{\alpha-1} \in L^p_{\text{loc}}$.

REFERENCES

1. W. Allegretto, *On the equivalence of two types of oscillation for elliptic operators*, Pacific J. Math. **55** (1974), 319–328.

2. B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (1981), 525–598.

3. J. Piepenbrink, *Nonoscillatory elliptic equations*, J. Differential Equations **15** (1974), 541–550.

4. M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

5. M. Reed and B. Simon, *Methods of modern mathematical physics*. IV: *Analysis of operators*, Academic Press, New York, 1978.

6. B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. **7** (1982), 447–526.

DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218

DEPARTMENTS OF MATHEMATICS AND PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125