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ON QUOTIENT RINGS OF TRIVIAL EXTENSIONS

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ABSTRACT. Let R be a ring with identity and M a two-sided R-module. It is shown that every right quotient ring in the sense of Gabriel of the trivial extension of R by M is a trivial extension of a right quotient ring of R by a suitable two-sided module in case ${}_RM$ is flat and finitely generated by elements which centralize with every element of R.

Let R be a ring with identity and M a two-sided R-module. The cartesian product $R \times M$ with componentwise addition and with multiplication given by $(r, m)(r', m') = (rr', rm' + mr')$ becomes a ring. This ring is called the trivial extension of R by M and denoted by $R \ltimes M$.

The purpose of this paper is to give a description of quotient rings of $R \ltimes M$. Here we shall use "quotient rings" in the sense of Gabriel [**2**]. In [**4**], Lambek showed that a maximal quotient ring of R is the biendomorphism ring of the injective envelope of the module R over itself. Later Morita proved in [**5**] that every quotient ring is the biendomorphism ring of a certain injective module. As every injective module V over $R \ltimes M$ is induced by a certain injective module U over R, that is, $V \cong \operatorname{Hom}_R(R \ltimes M, U)$, we shall first examine biendomorphism rings of modules over $R \ltimes M$ of the form $\operatorname{Hom}_R(R \ltimes M, X)$, and then investigate quotient rings of $R \ltimes M$, in particular, the maximal quotient ring of $R \ltimes M$ restricting ourselves to a special class of trivial extentions. Our main result of the present paper is stated as follows: If ${}_RM$ is flat and finitely generated by elements which centralize with every element of R, then every right quotient ring of $R \ltimes M$ is a trivial extension of a right quotient ring of R by a two-sided module. It is to be noted that the conclusion of the result is not necessarily valid without some hypothesis (see the example below).

Throughout this paper, R denotes a ring with identity, M a two-sided R-module and T the trivial extension of R by M. All modules are unital and module homomorphisms are written on the side opposite the scalars. The notation $r_Y(X)$ (resp. $l_Y(X)$) denotes the right (resp. left) annihilator of X in Y.

Let X be any right R-module. Then $\operatorname{Hom}_R(M, X)$ is a right R-module in a natural way: $(fa)(m) = f(am)$ for a in R, m in M and f in $\operatorname{Hom}_R(M, X)$. Moreover a direct sum $X \oplus \operatorname{Hom}_R(M, X)$ of right R-modules X and $\operatorname{Hom}_R(M, X)$ is a right T-module with multiplication given by $(x, f) \cdot (a, m) = (xa + f(m), f \cdot a)$, which is denoted by $H(X)$. $H(X)$ is nothing but the right T-module $\operatorname{Hom}_R(T, X)$. This fact will be employed frequently in the sequel.

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Let U and X be right R-modules. Following Morita [5], we say that U cogenerates X finitely provided that there exists a finite number of elements g_i in $\mathrm{Hom}_R(X, U)$ $(i = 1, \ldots, n)$ such that $\bigcap\{\mathrm{Ker}(g);\ g \in \mathrm{Hom}_R(X, U)\} = \bigcap\{\mathrm{Ker}(g_i);\ i = 1, \ldots, n\}$. In case U is injective, U cogenerates X finitely iff $\mathrm{Hom}_R(X, U)$ is finitely generated as a left $\mathrm{End}_R(U)$-module.

PROPOSITION 1. *Let U be a right R-module. Then $H(U)_T$ cogenerates T_T finitely if and only if U_R cogenerates M_R finitely and $H(U)_R$ cogenerates R_R finitely.*

PROOF. Assume $H(U)_T$ cogenerates T_T finitely and $\bigcap\{r_T(u_i, m_i^*);\ i = 1, \ldots, n\} = r_T(H(U))$ $(u_i \in U,\ m_i^* \in M^*)$, where $M^* = \mathrm{Hom}_R(M, U)$. Since $\bigcap\{\mathrm{Ker}(m_i^*);\ i = 1, \ldots, n\} = \bigcap\{\mathrm{Ker}(m^*);\ m^* \in M^*\}$ and $\bigcap\{r_R(u_i, m_i^*);\ i = 1, \ldots, n\} = r_R(H(U))$, U_R cogenerates M_R finitely and $H(U)_R$ does so to R_R.

Conversely, suppose $\bigcap\{r_R(u_i, m_i^*);\ i = 1, \ldots, n\} = r_R(H(U))$ $(u_i \in U,\ m_i^* \in M^*)$ and $\bigcap\{\mathrm{Ker}(g_j);\ j = 1, \ldots, s\} = \bigcap\{\mathrm{Ker}(m^*);\ m^* \in M^*\}$ $(g_j \in M^*)$. Setting $h_i = (u_i, m_i^*)$ $(1 \leqslant i \leqslant n)$ and $h_{n+j} = (0, g_j)$ $(1 \leqslant j \leqslant s)$, we have $\bigcap\{r_T(h_i);\ i = 1, \ldots, n + s\} = r_T(H(U))$.

Let U be a right R-module and $S = \mathrm{End}_R(U)$. Let $R' = \mathrm{BiEnd}_R(U)$ be the biendomorphism ring of U_R, that is, $R' = \mathrm{End}_S(U)$. If a ring homomorphism ρ_U of R to R' defined by $(u)(\rho_U(r)) = ur$ $(r \in R,\ u \in U)$ is surjective, then U_R is said to be balanced. For a right R-module X, X^* denotes its U-dual module $\mathrm{Hom}_R(X, U)$. X^* is a left S-module in a natural way and there is a mapping σ_X of X to its double dual module $X^{**} = \mathrm{Hom}_S(X^*, U)$ given by $(f)(\sigma_X(x)) = f(x)$ $(x \in X,\ f \in X^*)$. If σ_X is injective, then we say that U_R cogenerates X_R. Let us set $V = H(U)$, $U'' = \mathrm{Hom}_S(U, M^*)$ and $R'' = \mathrm{End}_S(M^*)$. Then the set $\left(\begin{smallmatrix} R' & U'' \\ M^{**} & R'' \end{smallmatrix}\right)$ consisting of all matrices $\left(\begin{smallmatrix} \alpha & \gamma \\ \beta & \delta \end{smallmatrix}\right)$ with α in R', β in M^{**}, γ in U'' and δ in R'' is a ring with the usual addition and multiplication of matrices given by

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} + \begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha + \alpha' & \gamma + \gamma' \\ \beta + \beta' & \delta + \delta' \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \cdot \begin{pmatrix} \alpha' & \gamma' \\ \beta' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + \gamma\beta' & \alpha\gamma' + \gamma\delta' \\ \beta\alpha' + \delta\beta' & \beta\gamma' + \delta\delta' \end{pmatrix}.$$

Noting $V = U \oplus M^*$ as an S-module, $\mathrm{End}_S(V) = \left(\begin{smallmatrix} R' & U'' \\ M^{**} & R'' \end{smallmatrix}\right)$. Setting $N = \mathrm{Hom}_R(M^*, U)$, we have by the proof of [6, Theorem 10] that $\mathrm{End}_T(V) \cong S \ltimes N$ in a natural way;

$$\begin{aligned} \mathrm{End}_T(V) = \mathrm{Hom}_T(V, \mathrm{Hom}_R(T, U)) &\cong \mathrm{Hom}_R(V \otimes_T T, U) \\ &\cong \mathrm{Hom}_R(V, U) \cong \mathrm{Hom}_R(U, U) \times N = S \times N. \end{aligned}$$

We shall identify $\mathrm{End}_T(V)$ with $S \ltimes N$. Then the module action of V as a left $S \ltimes N$-module is given by $(s, h) \cdot (u, m^*) = (su + h(m^*), s \cdot m^*)$ with (s, h) in $S \ltimes N$ and (u, m^*) in V. Since S can be imbedded in $S \ltimes N$ as a subring by a mapping $s \to (s, 0)$, $\mathrm{End}_{S \ltimes N}(V)$ is a subring of $\mathrm{End}_S(V)$. For convenience, a pair (α, δ) $(\alpha \in R',\ \delta \in R'')$ is called a (*)-pair provided that $(h(m^*))\alpha = h((m^*)\delta)$ for all h in N and all m^* in M^*. Then it is easy to see that $\mathrm{End}_{S \ltimes N}(V)$ consists of all elements $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ in $\begin{pmatrix} R' & U'' \\ M^{**} & R'' \end{pmatrix}$ such that (α, δ) is a (*)-pair, $P \subset \mathrm{Ker}(\gamma)$ and $\mathrm{Im}(\gamma) \subset W$, where $P = \Sigma\{\mathrm{Im}(h);\, h \in N\}$ and $W = \bigcap\{\mathrm{Ker}(h);\, h \in N\}$. Assume V_T is balanced. Then a ring homomorphism $\rho_V\colon T \to \mathrm{BiEnd}_T(V)$ given by

$$\rho_V(a, m) = \begin{pmatrix} \rho_U(a) & 0 \\ \sigma_M(m) & \rho_{M^*}(a) \end{pmatrix}$$

is surjective. Thus we have the following (a), (b) and (c).

(a) The natural homomorphism $\sigma_M\colon M \to M^{**}$ is surjective.

(b) If $(\alpha, \delta) \in R' \times R''$ is a (*)-pair, then there is some a in R with $\alpha = \rho_U(a)$ and $\delta = \rho_{M^*}(a)$.

(c) For γ in U'', if $P \subset \mathrm{Ker}(\gamma)$ and $\mathrm{Im}(\gamma) \subset W$, then $\gamma = 0$.

Conversely if (a), (b) and (c) hold, then V_T is balanced clearly. Hence we have the following.

PROPOSITION. 2. *Using the same notations as above*, $\mathrm{BiEnd}_T(H(U))$ *is the subring of* $\begin{pmatrix} R' & U'' \\ M^{**} & R'' \end{pmatrix}$ *consisting of all elements* $\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$ *such that* (α, δ) *is a (*)-pair*, $P \subset \mathrm{Ker}(\gamma)$ *and* $\mathrm{Im}(\gamma) \subset W$. $H(U)_T$ *is balanced if and only if* (a), (b) *and* (c) *are all valid.*

COROLLARY. *Assume U_R is balanced and cogenerates M^*_R. Then $H(U)_T$ is balanced if (and only if)* (a) *is valid.*

PROOF. Since U_R cogenerates M^*_R, we have $W = 0$. Thus (c) is obvious. Suppose (α, δ) is a (*)-pair. Since U_R is balanced, $\alpha = \rho_U(a)$ for some a in R. Then $h((m^*)\delta) = (h(m^*))\alpha = h(m^*)a = h(m^*a)$ for all h in N, which yields that $(m^*)\delta - m^*a \in W$, and so $\delta = \rho_{M^*}(a)$, proving (b).

LEMMA 3. *Assume $M = \Sigma_{i=1}^n Rm_i$ with $rm_i = m_i r$ for all r in R $(i = 1, \ldots, n)$.*

(a) *If U_R is injective, then there exist $h_1, \ldots, h_n$ in $N = \mathrm{Hom}_R(M^*, U)$ such that $N = \Sigma Sh_i$ with $sh_i = h_i s$ for all s in S $(i = 1, \ldots, n)$.*

(b) *If M^*_R is injective, then there exists uniquely a ring homomorphism $\lambda\colon R' \to R''$ such that $(\alpha, \lambda(\alpha))$ is a (*)-pair for every α in R'. Moreover M^{**} is a two-sided R'-module via λ; $\alpha \cdot x = \lambda(\alpha) \cdot x$ $(\alpha \in R', x \in M^{**})$.*

PROOF. (a) For each $i = 1, \ldots, n$ let h_i be a mapping of M^* to U given by $h_i(m^*) = m^*(m_i)$ for m^* in M^*. Then h_i is in N and $h_i s = sh_i$ for all s in S. A mapping η of M^* to $U^{(n)}$ defined by $\eta(m^*) = (h_i(m^*))_i$ is a left S- and right R-monomorphism, where $U^{(n)}$ denotes the direct sum of n-copies of U. This shows that $N = \Sigma Sh_i$ because U_R is injective.

(b) Since M^*_R is injective, the above mapping η is split as an R-map and so there are R-maps g_i of U to M^* $(i = 1, \ldots, n)$ such that $\Sigma g_i h_i = \mathrm{id}_{M^*}$. For each α in R', let $\lambda(\alpha)$ be a mapping of M^* to M^* given by $(m^*)\lambda(\alpha) = \Sigma g_i((h_i(m^*))\alpha)$. Since $h_j g_i$ is in S,

$$h_j((m^*)\lambda(\alpha)) = \sum_i (h_j g_i)((h_i(m^*))\alpha) = \left(h_j\left(\left(\sum_i g_i h_i\right) \cdot m^*\right)\right)\alpha = \left(h_j(m^*)\right)\alpha,$$

which yields by $h_j s = sh_j$ that $h_j((s\cdot m^*)\lambda(\alpha)) = (h_j(s\cdot m^*))\alpha = (sh_j(m^*))\alpha = s((h_j(m^*))\alpha) = sh_j((m^*)\lambda(\alpha)) = h_j(s\cdot(m^*)\lambda(\alpha))$. It follows that $\eta((s\cdot m^*)\lambda(\alpha)) = \eta(s\cdot(m^*)\lambda(\alpha))$, and so, $(s\cdot m^*)\lambda(\alpha) = s\cdot(m^*)\lambda(\alpha)$. Hence $\lambda(\alpha)$ is an S-map and λ is a mapping of R' to R''. Since h_i's generate N over S, $(\alpha, \lambda(\alpha))$ is a (*)-pair. If (α, δ) is a (*)-pair, then $\eta((m^*)\delta) = (h_i((m^*)\delta))_i = ((h_i(m^*))\alpha)_i = \eta((m^*)\lambda(\alpha))$, and so, $\delta = \lambda(\alpha)$. Thus λ is a unique ring homomorphism of R' to R'' with $(\alpha, \lambda(\alpha))$ a (*)-pair for every α, because $(\alpha\alpha', \lambda(\alpha)\lambda(\alpha'))$, $(\alpha+\alpha', \lambda(\alpha)+\lambda(\alpha'))$ and $(1, 1)$ are (*)-pairs. Noting M^{**} is an (R'', R')-bimodule in a natural way, M^{**} is a two-sided R'-module via λ, proving the lemma.

THEOREM 4. *Assume ${}_RM$ is flat and finitely generated by elements which centralize with every element of R. Then every right quotient ring of T is a trivial extension of a right quotient ring of R by a two-sided module.*

PROOF. Let Q be any right quotient ring of T. By Morita [**5**], Q is the biendomorphism ring of a suitable injective T-module V which cogenerates T_T finitely. Let $\overline{M} = \{(0, m) \in T;\, m \in M\}$ and $U = l_V(\overline{M})$. We shall first show that U is injective as a right R-module and $V \cong \mathrm{Hom}_R(T, U)$ as T-modules. Let I be any right ideal of R and $f\colon I \to U$ any R-map. The subset J of T consisting of all elements (a, m) with a in I and m in M is clearly a right ideal of T. Since a mapping g of J to V given by $g(a, m) = f(a)$ is a T-map and V is injective, there is some v in V such that $g(a, m) = v(a, m)$ for all (a, m) in J. Moreover v is in U and $f(a) = va$ for all a in I. It follows that U is injective. Thus there is an R-submodule U' of V such that $V = U \oplus U'$. Let p and p' be R-projections of V to U and U' respectively. The composition $\phi\colon V \to \mathrm{Hom}_R(T, U)$ of $\psi\colon V \to \mathrm{Hom}_R(T, V)$ and $\mathrm{Hom}(1, p)\colon \mathrm{Hom}_R(T, V) \to \mathrm{Hom}_R(T, U)$ is then a T-map, where ψ is defined by $\psi(v)(t) = v\cdot t$ for v in V and t in T. We claim ϕ is a bijection. Assume $v = u + u'$ $(u \in U, u' \in U')$ is in $\mathrm{Ker}(\phi)$. Since $v\cdot(a, m) = u\cdot(a, m) + u'\cdot(0, m) + u'\cdot(a, 0)$ and $u'\cdot(0, m) \in U$, we have $u\cdot(a, m) + u'\cdot(0, m) = \phi(v)(a, m) = 0$, which yields $u = 0$ and $u'\cdot(0, m) = 0$. Thus $v = u' \in U \cap U' = 0$, and so, ϕ is injective. Recalling $\mathrm{Hom}_R(T, U) = H(U)$, let (u, m^*) be any element in $H(U)$. Let $f\colon \overline{M} \to V$ be a T-map given by $f(0, m) = m^*(m)$. Since V is injective, there is some v_1 in V with $f(0, m) = v_1(0, m)$, that is, with $m^*(m) = v_1(0, m)$ for all m in M. Setting $u' = p'(v_1)$, $\phi(u + u') = (u, m^*)$, and so, ϕ is surjective. Hence ϕ is a bijection. We shall identify V with $H(U)$. Since U_R is injective and ${}_RM$ is flat, M_R^* is injective. Thus we can use the preceding lemma and its proof. In particular, M^{**} is a two-sided R'-module. With the same meanings of notation as mentioned previously, the fact that η is monic shows that $W = 0$, which yields from Proposition 2 and Lemma 3 that $\mathrm{BiEnd}_T(V) = \{\left(\begin{smallmatrix}\alpha & 0\\ \beta & \lambda(\alpha)\end{smallmatrix}\right);\, \alpha \in R', \beta \in M^{**}\}$ $(\cong R' \ltimes M^{**})$. Since U_R generates and cogenerates M_R^*, $R' = \mathrm{BiEnd}_R(U) \cong \mathrm{BiEnd}_R(U \oplus M^*)$ over R by [**1**, Lemma 14.1, p. 158]. But $(U \oplus M^*)_R$ cogenerates R_R finitely by Proposition 1 and is injective. Hence $\mathrm{BiEnd}_R(U \oplus M^*)$ and so R' is a right quotient ring of R. This proves the theorem.

REMARK. Under the same assumption as in Theorem 4, the proof shows that M^{**} is a two-sided R'-module and $\mathrm{BiEnd}_R(U) \ltimes M^{**}$ is a right quotient ring of T if U_R is injective such that U_R cogenerates M_R finitely and $(U \oplus M^*)_R$ does R_R finitely.

In the rest of this paper, L will denote the left annihilator of M in R, U the injective envelope of the right R-module $L \oplus M$, i: $M \to U$ the canonical injection and Q the maximal right quotient ring of T. We claim that $H(U)$ is an injective envelope of T_T. Let $K = \{(a, m) \in T;\ a \in L,\ m \in M\}$. Then $K = l_T(\overline{M})$ and K is essential in T as a right T-module. Let ψ: $K \to \mathrm{Hom}_R(T, K)$ be a mapping given by $\psi(k)(t) = k \cdot t$. Then ψ is clearly a T-monomorphism. $\mathrm{Im}(\psi)$ is further essential in $\mathrm{Hom}_R(T, K)$ $(= H(K))$, because if $(k, g) \in H(K)$ $(k \in K,\ g \in \mathrm{Hom}_R(M, K))$ is nonzero, then in case $g = 0$, $\psi(k) = (k, 0) \neq 0$ and in case $g \neq 0$, $g(m) \neq 0$ for some m in M, and so, $\psi(g(m)) = (k, g) \cdot (0, m) \neq 0$. Similarly, we can see that $H(K)$ is essential in $H(U)$. It follows that $H(U)$ is an injective envelope of T_T and $Q = \mathrm{BiEnd}_T(H(U))$. Hence we have the following from the proof of Theorem 4.

PROPOSITION 5. *Under the same assumption as in Theorem* 4, *Q is the trivial extension* $\mathrm{BiEnd}_R(U) \ltimes M^{**}$. *In particular, Q is T itself if and only if U_R is balanced and σ_M: $M \to M^{**}$ is surjective.*

PROPOSITION 6. *Assume $L = 0$. Then Q is right self-injective if and only if the following* (i) *and* (ii) *hold.*

(i) $U = \{h(i);\ h \in N\}$ *and* $h(i) = 0$ *implies* $h = 0$, *where* $N = \mathrm{Hom}_R(M^*, U)$. (ii) *$M^*$ is a free S-module with i as a basis. When this is the case, Q is a trivial extension.*

PROOF. As mentioned previously, $\mathrm{End}_T(H(U)) \cong S \ltimes N$ canonically and the canonical mapping ψ: $T \to H(U)$ defined by $\psi(a, m) = (i(m), i \cdot a)$ is monic and $\mathrm{Im}(\psi)$ is essential in $H(U)$. In particular, $H(U)$ is the injective envelope of T_T. By [**4**, Proposition 5], Q_Q is injective iff $\mathrm{End}_T(H(U)) \cong H(U)$ canonically, and in this case $Q \cong \mathrm{End}_T(H(U))$ canonically as rings. But the fact that $S \ltimes N \cong H(U)$ via $(s, h) \to (s, h) \cdot \psi(1, 0) = (h(i), s \cdot i)$ is evidently equivalent to (i) and (ii). This completes the proof.

A typical example of a two-sided module M described in Theorem 4 is a two-sided module $R^{(n)}$, the direct sum of n-copies of R. Let $M = R^{(n)}$. U is then the injective envelope $E(M_R)$ of M_R; $U = E(R_R)^{(n)}$. Hence $\mathrm{BiEnd}_R(U) \cong \mathrm{BiEnd}_R(E(R_R))$ over R. Therefore $Q = Q_{\max}(R) \ltimes M^{**}$ by Proposition 5, where $Q_{\max}(R)$ denotes the maximal right quotient ring of R. Assume now that R has no 2 torsion, that is, $2a = 0$ $(a \in R)$ implies $a = 0$. Then a ring automorphism g: $T \to T$ given by $g(a, m) = (a, -m)$ generates a finite group G of order $|G| = 2$ and R is the fixed subring $T^G = \{t \in T;\ g(t) = t$ for all g in $G\}$ of T relative to G. But every ring automorphism of a ring A can be extended uniquely to that of $Q_{\max}(A)$ [**7**, (1.17)]. Especially, G can be extended to a finite group of ring automorphisms of Q. Then $Q^G = Q_{\max}(R) = Q_{\max}(T^G)$. Thus the theorem gives a class of nonsemiprime rings A with finite group G of ring automorphisms and with no $|G|$ torsion such that $Q_{\max}(A)^G = Q_{\max}(A^G)$. For a semiprime ring A with a finite group G of automorphisms and with no $|G|$ torsion, it was proved by Kharchenko that A was Goldie iff A^G was Goldie and in this case $Q_{\mathrm{cl}}(A)^G = Q_{\mathrm{cl}}(A^G)$ [**3**]. Here $Q_{\mathrm{cl}}(A)$ denotes the classical quotient ring of A.

Finally the following example shows that the conclusion of Theorem 4 is not valid even if ${}_RM$ is finitely generated and projective.

EXAMPLE. Let k be a field and $R = k \times k$ the ring of direct product of k and k. Let $M = k$. Then M is a two-sided R-module with operations given by $(a, a') \cdot m = a'm$, $m \cdot (a, a') = ma$. Because $R \ltimes M$ is isomorphic to the ring $T_2(k)$ of lower triangular 2×2 matrices over k and the maximal right quotient ring of $T_2(k)$ is the ring of 2×2 matrices over k, that of $R \ltimes M$ is not a trivial extension by any nonzero two-sided module.

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