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WITH RESPECT TO TAIL SIGMA FIELDS, STANDARD MEASURES POSSESS MEASURABLE DISINTEGRATIONS

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ABSTRACT. Let P be a countably additive probability on a standard space, and let $\mathcal{Q}$ be a tail subfield. Though no disintegration of P with respect to $\mathcal{Q}$ that is countably additive need exist, there always is one which is finitely additive.

A surprising discovery of de Finetti, [**5**, p. 205] (see also [**4**]), leads to the conclusion that probability measures which are not countably additive need not possess any disintegration with respect to certain partitions. Even countably additive, fair-coin-tossing measure possesses no proper, measurable, countably additive disintegration with respect to the usual tail field [**3**]. One of the questions that then arises concerns the existence of *any* disintegration for this and similar situations. For standard measures and tail fields, an affirmative answer is provided by the theorem below. But first a little notation and two or three definitions are needed.

Throughout this note, $\mathcal{Q}$ is a subsigma field of a sigma field $\mathcal{B}$ of subsets of a set Ω, and P is a countably additive probability defined for $\mathcal{B}$. For each $\omega \in \Omega$, let $\mathcal{Q}(\omega)$ be the intersection of all A such that $\omega \in A \in \mathcal{Q}$. Whether or not $\mathcal{Q}(\omega) \in \mathcal{Q}$, call $\mathcal{Q}(\omega)$ an *$\mathcal{Q}$-atom*, and verify that the $\mathcal{Q}$-atoms constitute a partition $\Pi_{\mathcal{Q}}$ of Ω. A function σ of two variables, $h \in \Pi_{\mathcal{Q}}$, $B \in \mathcal{B}$ is a *measurable disintegration* of P given $\mathcal{Q}$ if:

(i) For each h, $\sigma(\cdot \mid h)$ is a probability on $\mathcal{B}$, possibly finitely additive, which is supported by h;

(ii) For each B, $\sigma(B \mid \mathcal{Q}(\omega))$ is a version of the usual P-conditional-expectation of B given $\mathcal{Q}$.

A *tail* field is the intersection of a decreasing sequence of subsigma fields of $\mathcal{B}$ each of which is countably generated.

THEOREM 1. *Let $(\Omega, \mathcal{B})$ be a standard space or, more generally, a Lusin space, and let $\mathcal{Q}$ be a tail subfield of $\mathcal{B}$. Then every countably additive probability, P, on $\mathcal{B}$ possesses a measurable disintegration given $\mathcal{Q}$.*

If the conclusion of Theorem 1 holds for $\mathcal{Q}$, call $\mathcal{Q}$ *tame*.

LEMMA 1. *Countably generated $\mathcal{Q}$'s in Lusin spaces are tame.*

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PROOF. In a stronger form, this assertion is contained in [**2**, Theorem 2] which, in turn, was seen as a simple corollary to [**1**, Theorem 5]. □

In view of Lemma 1, to establish Theorem 1, it suffices to establish

PROPOSITION 1. *The intersection of a decreasing sequence of sigma fields, each of which is tame, is tame.*

It is helpful to restate Proposition 1 in a more informative form, as in the next lemma, in which a diffuse mixture of disintegrations plays a role. A probability on the integers is *diffuse* if each finite set is of probability zero.

LEMMA 2. *Let P be a countably additive probability on $\mathfrak{B}$, $\mathcal{Q}$ the intersection of a decreasing sequence of tame fields $\mathcal{Q}_n$, σ_n a measurable disintegration of P given $\mathcal{Q}_n$, μ a diffuse probability on the positive integers, and σ_μ defined for $B \in \mathfrak{B}$, $\omega \in \Omega$ by*

$$\sigma_\mu(B \mid \mathcal{Q}(\omega)) = \int \sigma_n(B \mid \mathcal{Q}_n(\omega))\, \mu(dn). \tag{1}$$

Then σ_μ is a measurable disintegration of P given $\mathcal{Q}$.

The proof of Lemma 2 requires two preliminary facts.

SCHOLIUM 1. *Let $\mathcal{Q}$ be the intersection of sigma fields $\mathcal{Q}_1 \supset \mathcal{Q}_2 \supset \cdots$, and let $A_n \in \mathcal{Q}_n$. Then* $\liminf A_n \in \mathcal{Q}$.

PROOF. Let $B_j = \bigcap A_n$ $(n \geqslant j)$. Plainly, since $B_i \in \mathcal{Q}_i \subset \mathcal{Q}_j$ for $i > j$, $\bigcup B_i$ $(i \geqslant j)$ $\in \mathcal{Q}_j$. And since this union is $\liminf A_n$, independent of j, it belongs to every $\mathcal{Q}_j$, that is, it belongs to $\mathcal{Q}$. □

SCHOLIUM 2. *If $\mathcal{Q}$ is the intersection of sigma fields $\mathcal{Q}_1 \supset \mathcal{Q}_2 \supset \cdots$, then, for each $\omega \in \Omega$,*

$$\mathcal{Q}(\omega) = \bigcup \mathcal{Q}_n(\omega). \tag{2}$$

PROOF. Since $\mathcal{Q} \subset \mathcal{Q}_n$, $\mathcal{Q}(\omega) \supset \mathcal{Q}_n(\omega)$ for all n, the left-hand side of (2) includes the right-hand side. It suffices to see, therefore, that unless $\omega' \in \Omega$ belongs to one of the $\mathcal{Q}_n(\omega)$, it does not belong to $\mathcal{Q}(\omega)$. So assume that $\exists A_n \in \mathcal{Q}_n$ which separates ω from ω', say $\omega \in A_n$, $\omega' \in A_n^c$ for all n. Let $A = \liminf A_n$. Then A obviously separates ω from ω' and, by Scholium 1, $A \in \mathcal{Q}$. So $\omega' \notin \mathcal{Q}(\omega)$. □

PROOF OF LEMMA 2. If $\mathcal{Q}(\omega) = \mathcal{Q}(\omega')$, then, by Scholium 2, $\mathcal{Q}_n(\omega) = \mathcal{Q}_n(\omega')$ for some n and, hence, for all but a finite set of n. So, since μ is diffuse, $\sigma_\mu(B \mid \omega) = \sigma_\mu(B \mid \omega')$, that is, $\sigma_\mu(\cdot \mid \omega)$ depends only on $\mathcal{Q}(\omega)$, and may therefore be written as $\sigma_\mu(\cdot \mid \mathcal{Q}(\omega))$. Since $\sigma_n(\cdot \mid \mathcal{Q}_n(\omega))$ is supported by $\mathcal{Q}_n(\omega)$, and $\mathcal{Q}_n(\omega) \subset \mathcal{Q}(\omega)$, it is, for all n, certainly supported by $\mathcal{Q}(\omega)$.

Therefore, $\sigma_\mu(\cdot \mid \mathcal{Q}(\omega))$, too, is supported by $\mathcal{Q}(\omega)$. Hence, (i) is satisfied by σ_μ. To verify (ii), note that $\sigma_n(B \mid \mathcal{Q}_n(\omega))$, being a version of the P-conditional probability of B given $\mathcal{Q}_n$, converges, by the reversed martingale convergence theorem, for all ω not in a P-null set, to a limit $X(\omega)$, where X is a version of $P(B \mid \mathcal{Q})$. Since μ is a diffuse probability, for all such ω, the right-hand side of (1) is $X(\omega)$. So $\sigma_\mu(B \mid \mathcal{Q}(\omega))$ satisfies (ii). □

Plainly, Theorem 1 implies that countably additive probabilities possess measurable disintegrations given the usual tail fields as well as fields of the form $\mathcal{F}_{t+}$ familiar in the usual theory of continuous-path processes. The same conclusion holds for the subfield of Borel sets invariant under a group which is the union of an increasing sequence of finite groups as is, for example, the group of periodic rotations of a circle. In particular, the usual fair-coin probability possesses a measurable disintegration given the tail sigma-field which, at the time [3] was written, seemed unlikely. Since, as reported there, all such disintegrations are purely finitely additive, the conclusion of Theorem 1 cannot be strengthened to assert the existence of measurable disintegrations which are countably additive. That the hypothesis that P is countably additive cannot be dropped can be seen by means of an example [**4**, Theorem 2].

Whether the conclusion of Theorem 1 holds if $\mathcal{Q}$ consists of the Borel subsets of the unit circle invariant under a nonperiodic rotation, we do not know. Indeed, whether the hypothesis of Theorem 1, that $\mathcal{Q}$ be a tail field, can be considerably weakened, if not entirely eliminated, we must leave open. To us,this seems dubious even if "measurable" is deleted from the conclusion.

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