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ESSENTIALLY SUBNORMAL OPERATORS AND K-SPECTRAL SETS

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ABSTRACT. Let T be an essentially subnormal operator. We give six conditions which are equivalent to the spectrum of T being a K-spectral set. From this follow two corollaries which give sufficient conditions for invariant subspaces of essentially subnormal operators. Several examples are given that show that some essentially subnormal operators are not essentially normal nor perturbations of subnormal operators.

1. Introduction. In [7] J. G. Stampfli proved that every bounded linear operator on a (separable) Hilbert space whose spectrum is a K-spectral set has a nontrivial invariant subspace. As a corollary he proved that every essentially normal operator satisfying a certain boundedness condition (see assertion (2) in Theorem 2 below) necessarily has a nontrivial invariant subspace.

Corollary 1 below extends Stampfli's corollary to the class of essentially *subnormal* operators. Our corollary itself follows from Theorem 2, which gives for an essentially subnormal operator six conditions equivalent to its spectrum being a K-spectral set. These results appear in §2; in §3 we give examples and related results.

2. Main results. Let T be a bounded linear operator on the complex separable Hilbert space H. Denote its spectrum by $\sigma(T)$, and let R_T be the uniform closure of the set of rational functions in T with poles outside $\sigma(T)$. Recall that $\sigma(T)$ is a K-spectral set (for T) [7] if there exists $K > 0$ such that, for each $f(T) \in R_T$,

$$\| f(T) \| \leqslant K \| f \|_{\sigma(T)}$$

where $\| f \|_{\sigma(T)}$ denotes the sup-norm of the scalar-valued function f on $\sigma(T)$. Given this, we can write

THEOREM 1 (STAMPFLI [7]). *Let T be an operator on H whose spectrum is a K-spectral set. Then T has a nontrivial invariant subspace.*

To state Theorem 2 we use the following notations. Let $L(H)$, $C(H)$ and $L(H)/C(H)$ denote, respectively, the algebra of bounded operators, the ideal of compact operators and the Calkin algebra on H. Let π: $L(H) \to L(H)/C(H)$ be the natural surjection. Then T is *essentially subnormal* if $\pi(T)$ is subnormal. This is

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interpreted as follows. Calkin proved [2] that for each $T \in L(H)$ the image $\pi(T)$ may be realized as an operator on some Hilbert space H_1 (the representation space). Thus, if $\pi(T)$ is subnormal, there is some normal operator N on an extension space $H_2 \supset H_1$ such that $N \mid H_1 = \pi(T)$. We note that every essentially normal operator is essentially subnormal.

Let $C_1(H)$ be the set of operators in $C(H)$ of unit norm, and let $\operatorname{dist}(R_T, C_1(H))$ be the distance from R_T to $C_1(H)$, i.e. $\inf\{\| f(T) - L\|: f(T) \in R_T, L \in C_1(H)\}$. We use g: $R_T \to C(\sigma(T))$ to denote the Gelfand transform into the continuous functions on $\sigma(T)$.

THEOREM 2. *Let T be essentially subnormal such that* (a)$R_T \cap C(H) = (0)$ *and* (b) $\sigma(T)$ *has no isolated points. Then the following assertions are equivalent*:

(1) $\sigma(T)$ *is a K-spectral set*;

(2) $\operatorname{dist}(R_T, C_1(H)) > 0$;

(3) π *is bounded below on* R_T;

(4) $R_T + C(H)$ *is uniformly closed in* $L(H)$;

(5) $\pi(R_T)$ *is closed in the Calkin algebra*;

(6) $g(R_T)$ *is closed in* $C(\sigma(T))$.

Moreover, if $T = S + L$, where S is subnormal and $L \in C(H)$ and $\sigma(T) = \sigma(S)$, then (1)–(6) *are equivalent to*

(7) $R_T + C(H) = R_S + C(H)$.

PROOF. We observe first that (4) and (5) are always equivalent. For if $R_T + C(H)$ is closed in $L(H)$, then $\pi(R_T) = R_T + C(H)/C(H)$ is a closed subalgebra of the Calkin algebra; and conversely, if $\pi(R_T)$ is closed then $R_T + C(H) = \pi^{-1}(\pi(R_T))$ is closed because π is continuous.

Statements (2) and (3) are also always equivalent. Suppose that (2) holds and that $\{f_n(T)\}$ is a sequence in R_T such that $\| f_n(T)\| = 1$ and $\|\pi(f_n(T))\| \to 0$. For each n there is some $L_n \in C(H)$ such that $\| f_n(T) - L_n\| \to 0$. Thus $\| L_n\| \to 1$, so it is clear that (2) is contradicted. This proves (2) $\Rightarrow$ (3), and the converse is proved by reversing the preceding argument.

Since (3) $\Rightarrow$ (5) is obvious, we prove (4) $\Rightarrow$ (3). From the hypothesis $R_T \cap C(H) = (0)$ and the assumption that $R_T + C(H)$ is closed, it follows that $\pi(R_T)$ is topologically isomorphic to R_T. Hence π is bounded below on R_T by the closed graph theorem. We now have (2)–(5) equivalent.

Next we prove that (3) $\Rightarrow$ (1). Let f be a (scalar) rational function with poles off $\sigma(T)$. If (3) holds, then there is some $K > 0$ (independent of f) such that $\| f(T)\| \leqslant K\|\pi(f(T))\|$. But $\sigma(\pi(T)) \subset \sigma(T)$, so $f(\pi(T))$ is defined and $f(\pi(T)) = \pi(f(T))$ since π is a homomorphism. It follows that

$$\|\pi(f(T))\| = \| f(\pi(T))\| \leqslant \| f \|_{\sigma(\pi(T))} \leqslant \| f \|_{\sigma(T)}$$

because $f(\pi(T))$ is subnormal and subnormal operators have spectra which are 1-spectral sets and the last inequality is obvious. Combining the inequalities above, one sees that $\sigma(T)$ is a K-spectral set for T.

To prove that (1) $\Rightarrow$ (3), suppose that $\sigma(T)$ is a K-spectral set and suppose there is a sequence $\{f_n(T)\}$ in R_T such that $\|f_n(T)\| = 1$ for each n and $\|\pi(f_n(T))\| \to 0$. Since $f_n(T)$ is essentially subnormal (each n),

$$\|\pi(f_n(T))\| = \sup\{|\lambda| : \lambda \in \sigma(\pi(f_n(T)))\}.$$

By [**4**, Theorem 2.4], the Weyl spectrum $w(f_n(T))$ consists of $\sigma(\pi(f_n(T)))$ (= the essential spectrum of $f_n(T)$) and some "holes" of the latter (bounded components of the complement). On the other hand, by [**6**, Theorem 4], for each n there is an $L_n \in C(H)$ such that $\sigma(f_n(T) + L_n) = w(f_n(T))$. But by Weyl's theorem (on perturbations) $\sigma(f_n(T))$ consists of $\sigma(f_n(T) + L_n)$ together with holes of the latter set and at most countably many isolated points. By hypothesis (b) and the spectral mapping theorem, $\sigma(f_n(T))$ has no isolated points. It follows that $\sigma(f_n(T))$ is formed from $\sigma(\pi(f_n(T)))$ by "filling in holes". Thus,

$$\sup\{|\lambda| : \lambda \in \sigma(\pi(f_n(T)))\} = \sup\{|\lambda| : \lambda \in \sigma(f_n(T))\} = \|f_n\|_{\sigma(T)},$$

and since $\sigma(T)$ is K-spectral we derive the contradiction

$$1 = \|f_n(T)\| \leqslant K\|f_n\|_{\sigma(T)} = K\|\pi(f_n(T))\| \to 0.$$

Hence π is bounded below on R_T.

Now we prove that hypothesis (a) implies that the Gelfand transform g is injective on R_T. If $g(L) = 0$ for some $L \in R_T$, then L is quasinilpotent ($\sigma(L) = \{0\}$). We show $L \in C(H)$. Let $\{f_n\}$ be a sequence of rational functions with poles off $\sigma(T)$ such that $f_n(T) \to L$. Put $S = \pi(T)$ and $Q = \pi(L)$. Then $f_n(S)$ is defined for each n, Q is quasinilpotent and $f_n(S) \to Q$. Let N be the (minimal) normal extension of the subnormal operator S, so $f_n(N)$ is defined. Moreover, $f_n(N) - f_j(N)$ is the normal extension of $f_n(S) - f_j(S)$ [**5**, Theorem 4.1], hence

$$\|f_n(N) - f_j(N)\| = \|f_n(S) - f_j(S)\| \to 0 \qquad (n, j \to \infty).$$

The Cauchy sequence $\{f_n(N)\}$ has limit $V \in L(H_2)$ (see the definition above), and V is clearly normal. Since $Q = V|H_1$ is thus subnormal, we must have $Q = 0$. This implies $L \in C(H)$, so that $L = 0$ and g is injective.

We can now easily prove the equivalence of (1) and (6). By definition of g we know that $\|g(f(T))\| = \|f\|_{\sigma(T)}$. If (6) holds, then R_T and $g(R_T)$ are topologically isomorphic and there is a $K > 0$ such that $\|f(T)\| \leqslant K\|g(f(T))\|$, so $\sigma(T)$ is K-spectral. Conversely, (1) implies that g is bounded below on R_T, so its image is closed. This completes the proof that (1)–(6) are equivalent.

Finally, suppose that $T = S + L$ with S subnormal and L compact such that $\sigma(T) = \sigma(S)$. For each rational f with poles off $\sigma(T)$, both $f(T)$ and $f(S)$ exist and $f(S) - f(T) \in C(H)$. For subnormal S, $R_S + C(H)$ is closed in $L(H)$ (apply (1) $\Rightarrow$ (4) to S), so (7) $\Rightarrow$ (4). Moreover, if $R_S + C(H)$ is closed, $R_T + C(H) \subset R_S + C(H)$. If $R_T + C(H)$ is closed, then the reverse inclusion holds, so (4) $\Rightarrow$ (7). This completes the proof of the theorem.

COROLLARY 1. *If T is essentially subnormal and $R_T + C(H)$ is closed, then T has a nontrivial invariant subspace.*

PROOF. If $\sigma(T)$ has an isolated point, then the Riesz-Dunford functional calculus yields a nontrivial invariant subspace. If $R_T \cap C(H) \neq (0)$, then R_T contains a compact operator commuting with T. By Lomonosov's lemma T has a nontrivial invariant subspace. Otherwise, $\sigma(T)$ is K-spectral by Theorem 2, hence T has a nontrivial invariant subspace by Theorem 1.

COROLLARY 2. *Let T be essentially subnormal and let g be the Gelfand transform on R_T. Then T has a nontrivial invariant subspace if either* $\ker g \neq 0$ *or $g(R_T)$ is closed.*

PROOF. If $\ker g \neq 0$, then by the proof of the injectivity of g in Theorem 2 it is clear that each L in $\ker g$ is compact. Hence the result follows in this case by Lomonosov's lemma. If $\ker g = 0$ and $g(R_T)$ is closed, then $\sigma(T)$ is K-spectral and the conclusion follows from Theorem 1.

3. Examples. In this section we consider several examples.

EXAMPLE A. The spectrum is not a K-spectral set for every essentially subnormal operator. Davidson and Fong [**3**] give an example of a compact perturbation $T = V + L$ of the bilateral shift V such that $\sigma(T) = \sigma(V)$ (= unit circle) such that $\|T^n\|$ is unbounded. Hence $\sigma(T)$ is not K-spectral for any $K > 0$.

EXAMPLE B. If some of the hypotheses of Theorem 2 fail, then the equivalence of (1)–(6) may also fail. If T is a compact normal operator ($\sigma(T) \neq \{0\}$), then (1) holds but (3) fails.

EXAMPLE C. An essentially subnormal operator need not be essentially normal nor subnormal and compact. Let A be a shift of infinite multiplicity, let B be a shift of multiplicity one, and let $T = A \oplus (B^* + 3)$. Now T is not essentially normal since $A^*A = I$ but AA^* is a projection of infinite corank. If T were a perturbation of a subnormal operator, then $\operatorname{ind}(\lambda - T) \leqslant 0$ for all $\lambda \notin \sigma(\pi(T))$. But clearly $\lambda = 3$ lies outside $\sigma(\pi(T)) = \{z: |z| \leqslant 1\} \cup \{z: |z - 3| = 1\}$, while $\operatorname{ind}(3 - T) = \operatorname{ind}(3 - A) + \operatorname{ind}(-B^*) = 0 + 1 = 1$.

The corollaries of §2 and the last example lead to the general question whether an essentially subnormal operator T has a nontrivial invariant subspace. If T is also essentially normal, then the question reduces (by the Brown-Douglas-Fillmore theory [**1**]) to that of $T = N + K$ (N normal, K compact) having an invariant subspace. On the other hand, since some essentially subnormal operators are not essentially normal (Example C), the solution to the case $T = N + K$ would still leave the question open in general.

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