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ON THE MAPPING TORUS OF AN AUTOMORPHISM

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ABSTRACT. Let ρ be an automorphism of a C^*-algebra A. The mapping torus $T_\rho(A)$ is the C^*-algebra of A-valued continuous functions x on $[0, 1]$ satisfying $x(1) = \rho(x(0))$. Using his Thom isomorphism theorem, A. Connes has shown that the K-groups of $T_\rho(A)$, with indices reversed, are isomorphic to those of the crossed product $A \times_\rho Z$. We provide here an alternative proof of this fact which gives an explicit description of the isomorphism.

Given a C^*-algebra A and an automorphism ρ of A, the mapping torus of the pair (A, ρ) is defined by

$$T_\rho(A) = \{x \in C(I, A): x(1) = \rho(x(0))\},$$

where I is the unit interval and $C(I, A)$ is the C^*-algebra of continuous A-valued functions on I. In **[1]**, A. Connes shows that the K-groups, with indices reversed, of $T_\rho(A)$ coincide with those of $A \times_\rho Z$ (the crossed product of A by the action of Z generated by ρ). He does this in order to obtain the exact sequence of M. Pimsner and D. Voiculescu **[3]** for $K_\#(A \times_\rho Z)$ as a consequence of the Thom isomorphism in **[1]**. In the present note, we give an alternative proof of the isomorphism of $K_{1-j}(T_\rho(A))$ with $K_j(A \times_\rho Z)$ which proceeds in somewhat the reverse fashion, namely starting from the sequence in **[3]**. One merit of this approach is that the isomorphism in question is described quite explicitly in terms of the elementary ingredients of K-theory.

We begin by recalling some pertinent facts and establishing notation. The crossed product $A \times_\rho Z$ is generated by terms of the form aL^n ($a \in A, n \in Z$), where $L = L_\rho$ is a unitary on the space of a certain representation of A satisfying $LaL^* = \rho(a)$ $\forall a \in A$. It will be convenient to assume that A is unital throughout most of what follows, a restriction that is easily removed at the appropriate time. For such A, we have $L \in A \times_\rho Z$. The exact sequence

$$(1)\qquad \begin{array}{ccccc} K_0(A) & \xrightarrow{\rho_* - \mathrm{id}} & K_0(A) & \xrightarrow{i_*} & K_0(A \times_\rho Z) \\ \uparrow & & & & \downarrow \\ K_1(A \times_\rho Z) & \xleftarrow[i_*]{} & K_1(A) & \xleftarrow[\rho_* - \mathrm{id}]{} & K_1(A) \end{array}$$

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was first obtained in [3] and subsequently in [2] and [1]. Here $i: A \to A \times_\rho Z$ is the natural inclusion, and the vertical arrows represent boundary maps to be discussed presently. In [3], the sequence above comes from an extension

$$0 \to A \otimes \mathcal{K} \to \mathcal{T} \to A \times_\rho Z \to 0 \tag{2}$$

of $A \times_\rho Z$, called the Toeplitz extension. To construct $\mathcal{T}$, represent $A \times_\rho Z$ on a Hilbert space H, and let U_+ be the unilateral shift on $l^2(Z^+)$. The Toeplitz algebra $\mathcal{T}$ is then the C^*-algebra on $H \otimes l^2(Z^+)$ generated by $A \otimes 1$ and the isometry $L \otimes U_+$. It is shown in [3] that there is a homomorphism $\pi: \mathcal{T} \to A \times_\rho Z$, taking $a \otimes 1$ to a and $L \otimes U_+$ to L, whose kernel is isomorphic to $A \otimes \mathcal{K}$. Most of the hard work in [3] consists in proving that the injection $a \mapsto a \otimes 1$ of A into $\mathcal{T}$ induces an isomorphism of K-groups. Identifying $K_\#(\mathcal{T})$ with $K_\#(A)$ in this manner transforms the K-theory exact sequence (§10 of [4]) for (2) into the sequence (1). In particular, if u is a unitary in $A \times_\rho Z$ that lifts to a partial isometry V in $\mathcal{T}$, the boundary map $\partial: K_1(A \times_\rho Z) \to K_0(A)$ takes the class of u to the index of V computed in $K_0(A \otimes \mathcal{K}) \approx K_0(A)$.

Consider now the map $e: T_\rho(A) \to A$ defined by $e(x) = x(0)$. The kernel of e is the reduced suspension SA $(= \{x \in C(I, A): x(0) = 0 = x(1)\})$, so, via the Bott maps that identify $K_j(SA)$ with $K_{1-j}(A)$, we have an exact sequence

$$\begin{array}{ccccc} K_1(A) & \to & K_0(T_\rho(A)) & \xrightarrow{e_*} & K_0(A) \\ \uparrow & & & & \downarrow \\ K_1(A) & \underset{e_*}{\leftarrow} & K_1(T_\rho(A)) & \leftarrow & K_0(A) \end{array} \tag{3}$$

It is the following lemma that suggests a relationship between (3) and (1).

LEMMA 1. *The vertical arrows in* (3) *both represent* $\rho_* - \mathrm{id}$.

PROOF. There is no loss of generality in working with unitaries and projections in A rather than in $A \otimes M_n$, since $T_{\rho\otimes \mathrm{id}_n}(A \otimes M_n)$ is isomorphic to $T_\rho(A) \otimes M_n$. For the K_0 arrow, let p be a projection on A and $[p]$ its class in $K_0(A)$. Define $x \in T_\rho(A)$ by $x(t) = (1-t)p + t\rho(p)$, so $e(x) = p$. The map $\delta: K_0(A) \to K_1(SA)$ from $SA \to T_\rho(A) \to A$ then takes $[p]$ to $[e^{2\pi i x}] \in K_1(SA^+) = K_1(SA)$ (where $^+$ denotes adjunction of a unit). But if $\beta: K_0(A) \to K_1(SA)$ is the Bott map, we also have $[e^{2\pi i x}] = \beta[\rho(p)] - \beta[p]$. This takes care of the K_0 case. One can give a similar, but somewhat more cumbersome, direct argument for K_1, or else proceed by replacing A with $A \otimes C(T)$, ρ with $\tilde{\rho} = \rho \otimes \mathrm{id}_T$, and identifying $T_{\tilde{\rho}}(A \otimes C(T))$ with $T_\rho(A) \otimes C(T)$. Write $\nu: K_1(A) \to K_1(A)$ for the left-hand vertical arrow in (3). When we decompose $K_0(A \otimes C(T))$ as $K_0(A) \oplus K_1(A)$, the right-hand arrow in the $A \otimes C(T)$ version of (3) is, on the one hand, $(\rho_* - \mathrm{id}) \oplus \nu$, and on the other, $\tilde{\rho}_* - \mathrm{id} = (\rho_* - \mathrm{id}) \oplus (\rho_* - \mathrm{id})$, so $\nu = \rho_* - \mathrm{id}$.

By the lemma, we have two parallel exact sequences:

$$
\text{(4)}\qquad
\begin{tikzcd}
 & & K_0(T_\rho(A)) \arrow[dr, "e_*"] & & \\
\rightarrow K_1(A) \arrow[r, "\rho_* - \mathrm{id}"'] & K_1(A) \arrow[ur, "\alpha"] \arrow[dr, "i_*"'] & & K_0(A) \arrow[r, "\rho_* - \mathrm{id}"'] & K_0(A) \rightarrow \\
 & & K_1(A \times_\rho Z) \arrow[ur, "\partial"'] & &
\end{tikzcd}
$$

where α is the composition of the Bott map $K_1(A) \to K_0(SA)$ with the map induced by the inclusion of SA into $T_\rho(A)$. What we need is a homomorphism γ: $K_0(T_\rho A) \to K_1(A \times_\rho Z)$ satisfying $\gamma\alpha = i_*$ and $\partial\gamma = e_*$. Such a map will automatically be an isomorphism by exactness of the two sequences.

The following lemma, needed in the definition of γ, is a variant of the familiar fact that norm-close projections in a unital C^*-algebra are unitarily equivalent.

LEMMA 2. *Let $\{p_t\colon t \in I\}$ be a path of projections in A. There is a path $\{w_t\}$ of unitaries in A with $w_0 = 1$ and $p_t = w_t p_0 w_t^* \ \forall t \in I$.*

PROOF. Fix $t_0 \in I$. There is an $\varepsilon > 0$ (independent of t_0) such that whenever $|t - t_0| < \varepsilon$, we have $\| p_t - p_{t_0} \| < 1$ and $\| r_t - p_t \| < 1$, where r_t is the idempotent $(1 - p_{t_0} + p_t)^{-1} p_{t_0} (1 - p_{t_0} + p_t)$. Notice that for such t, we have $r_t = r_t p_t$. Let $x_t = (1 - p_t + r_t)^{-1}(1 - p_{t_0} + p_t)^{-1}$. Then, easily, $x_t^{-1} p_t = p_{t_0} x_t^{-1}$ for t within ε of t_0. Partitioning I into subintervals of length less than ε, we can chain together the partial paths $\{x_t\}$ to obtain a full path $\{y_t\}$ of invertibles, with $y_0 = 1$, such that $p_t = y_t p_0 y_t^{-1}$. Since $|y_t|$ commutes with p_0 for each t, we may replace $\{y_t\}$ by the unitary path $\{w_t\} = \{y_t |y_t|^{-1}\}$.

Now we can move on to the main result.

THEOREM (CONNES [**1**]). *For a C^*-algebra A and an automorphism ρ of A, the groups $K_j(A \times_\rho Z)$ and $K_{1-j}(T_\rho(A))$ are isomorphic ($j = 0, 1$).*

PROOF. We will assume that A is unital for most of the argument, and deal directly just with the case $j = 1$. (The other isomorphism will be described more or less concretely in a separate remark.) Let p be a projection in $T_\rho(A)$ and let $\{w_t\}$ be an implementing path for $\{p(t)\}$ as in Lemma 2. In $A \times_\rho Z$ we have $Lp(0)L^* = \rho(p(0)) = p(1) = w_1 p(0) w_1^*$, so $L^* w_1$ commutes with $p(0)$. Thus $L^* w_1 p(0) + 1 - p(0)$, which we shall denote temporarily by $\gamma_0(p)$, is a unitary in $A \times_\rho Z$. If $\{v_t\}$ is another implementing path for $\{p(t)\}$, then $w_t^* v_t$ commutes with $p(0)$ for each t and $t \mapsto L^* w_1 w_t^* v_t p(0) + 1 - p(0)$ is a path of unitaries in $A \times_\rho Z$ joining $L^* w_1 p(0) + 1 - p(0)$ to $L^* v_1 p(0) + 1 - p(0)$. Thus the class $[\gamma_0(p)]$ in $K_1(A \times_\rho Z)$ is independent of the choice of implementing path. Further, if q is a projection in $T_\rho(A)$ unitarily equivalent to p, say $q = upu^*$ for some unitary $u \in T_\rho(A)$, then

$\{u(t)w_t u(0)^*\}$ is an implementing path for $\{q(t)\}$. In $K_1(A \times_\rho Z)$ we have

$$\begin{aligned}[\gamma_0(q)] &= [L^* u(1)w_1 u(0)^* q(0) + 1 - q(0)] \\ &= [L^*\rho(u(0)^*)u(1)w_1 p(0) + 1 - p(0)] = [\gamma_0(p)]\end{aligned}$$

(conjugating by $u(0)^*$ and using $\rho(u(0)) = u(1) = Lu(0)L^*$). Replacing A by $A \otimes M_n$ and ρ by $\rho \otimes \mathrm{id}_n$, we can define $\gamma_0(p)$ in $(A \times_\rho Z) \otimes M_n$ for a projection p in $A \otimes M_n$. Set $\gamma[p] = [\gamma_0(p)]$. The resulting map $\gamma\colon K_0(T_\rho(A)) \to K_1(A \times_\rho Z)$ is well defined by what has been observed above and obviously a homomorphism. We next check that $\gamma\alpha = i_*$. Let u be a unitary in (without loss of generality) A and let $\{v_t\}$ be a path of unitaries in $A \otimes M_2$ with $v_0 = 1 \oplus 1$ and $v_1 = u \oplus u^*$. Define $p \in T_\rho(A) \otimes M_2$ by $p(t) = v_t(1 \oplus 0)v_t^*$. Then $\alpha[u] = [p] - [1]$ (see §8 of **[4]**) and

$$\gamma([p]) = [(L^* \oplus L^*)(u \oplus u^*)(1 \oplus 0) + (0 \oplus 1)] = [L^* u].$$

Since $\gamma[1] = [L^*]$, we have $\gamma\alpha[u] = i_*[u]$. To see that $\partial\gamma = e_*$, let p be a projection in $T_\rho(A)$, with $\{w_t\}$ an implementing path for $\{p(t)\}$. We lift $\gamma_0(p)$ to $V \in \mathfrak{T}$ defined by

$$V = (L^* \otimes U_+^*)(w_1 \otimes 1)(p(0) \otimes 1) + (1 - p(0)) \otimes 1.$$

We have

$$V^*V = p(0) \otimes U_+ U_+^* + (1 - p(0)) \otimes 1 = 1 \otimes 1 - p(0) \otimes (1 - U_+ U_+^*),$$

and $VV^* = 1 \otimes 1$, so $\partial[\gamma_0(p)] = [p(0)] = e_*[p]$ as required. It follows that γ is an isomorphism.

Suppose now that A is not unital. Adjoining a unit, we obtain A^+ and its automorphism ρ^+. The exact sequence $0 \to A \to A^+ \to \mathbf{C} \to 0$ gives rise to exact sequences

$$\begin{gathered} 0 \to T_\rho(A) \to T_\rho(A^+) \to C(T) \to 0, \\ 0 \to A \times_\rho Z \to A^+ \times_{\rho^+} Z \to C(T) \to 0 \end{gathered}$$

in which the maps onto $C(T)$ admit right inverses. There are thus natural isomorphisms

$$K_0(T_\rho(A^+)) \approx K_0(T_\rho(A)) \oplus Z, \quad K_1(A^+ \times_{\rho^+} Z) \approx K_1(A \times_\rho Z) \oplus Z.$$

The isomorphism $\gamma^+\colon K_0(T_\rho(A^+)) \to K_1(A^+ \times_{\rho^+} Z)$ takes $[1]$ to $[L_{\rho^+}^*]$, so it respects these direct sum decompositions and restricts to an isomorphism of $K_0(T_\rho(A))$ with $K_1(A \times_\rho Z)$.

Finally, replacing A by $A \otimes C(T)$ and ρ by $\rho \otimes \mathrm{id}_T$, we have the isomorphism $\tilde\gamma$: $K_0(T_\rho(A) \otimes C(T)) \to K_1((A \times_\rho Z) \otimes C(T))$. If λ and μ denote the natural inclusions of $T_\rho(A)$ and $A \times_\rho Z$ into their tensor products with $C(T)$, it is clear that $\tilde\gamma\lambda_* = \mu_*\gamma$. Thus, when we decompose $K_0(T_\rho(A) \otimes C(T))$ as $K_0(T_\rho(A)) \oplus K_1(T_\rho(A))$, and likewise for $K_1((A \times_\rho Z) \otimes C(T))$, $\tilde\gamma$ maps $K_0(T_\rho(A))$ isomorphically onto $K_1(A \times_\rho Z)$. Hence the remaining direct summands must be isomorphic; the restriction of $\tilde\gamma$ to $K_1(T_\rho(A))$ is the desired K_1-to-K_0 map.

In general, ρ_* and $K_\#(A)$ do not determine $K_\#(A \times_\rho Z)$ unambiguously. However, an easy consequence of the theorem is that $K_\#(A \times_\rho Z)$ depends on ρ only up to homotopy.

COROLLARY. *Suppose that ρ_0 and ρ_1 are automorphisms of A joined by a (point-norm continuous) path $\{\rho_t\}$ of automorphisms. Then $K_\#(A \times_{\rho_0} Z)$ and $K_\#(A \times_{\rho_1} Z)$ are isomorphic.*

PROOF. For $x \in T_{\rho_0}(A)$ define $\theta x\colon I \to A$ by $(\theta x)(t) = \rho_t\rho_0^{-1}(x(t))$. The inequality

$$\|(\theta x)(t) - (\theta x)(s)\| \leqslant \|\rho_t\rho_0^{-1}(x(t)) - \rho_s\rho_0^{-1}(x(t))\| + \|x(t) - x(s)\|$$

shows that θx is continuous, and we have $(\theta x)(0) = x(0)$, $(\theta x)(1) = \rho_1\rho_0^{-1}(x(1)) = \rho_1((\theta x)(0))$. Thus, θ is an isomorphism of $T_{\rho_0}(A)$ with $T_{\rho_1}(A)$.

In conclusion, we remark that an isomorphism of $K_1(T_\rho(A))$ with $K_0(A \times_\rho Z)$ can be defined formulaically as follows. (For simplicity assume that A is unital.) Let H be a selfadjoint 2×2 scalar matrix with $e^{iH} = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. Given a unitary $u \in T_\rho(A)$, define a unitary $V_u \in S(A \times_\rho Z)^+ \otimes M_2$ by

$$V_u(t) = (1 \oplus L)e^{itH}(\rho^{-1}(u(1-t))^* \oplus 1)e^{-itH}(1 \oplus L^*)e^{itH}(u(0) \oplus 1)e^{-itH}$$

and analogously for unitaries in $T_\rho(A) \otimes M_n$. It is clear that $[u] \to [V_u]$ defines a homomorphism from $K_1(T_\rho(A))$ to $K_1(S(A \times_\rho Z))$. Composition of this with the Bott isomorphism $K_1(S(A \times_\rho Z)) \to K_0(A \times_\rho Z)$ gives a homomorphism which can be shown to compose correctly with the appropriate maps in the index-reversed version of (4) and is thus an isomorphism. We omit details.

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