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A NOTE ON POLYNOMIAL OPERATOR APPROXIMATION

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ABSTRACT. An example is given of an operator T contained in a block-diagonal algebra of operators $\mathcal{A}$, an ideal $J \subset \mathcal{A}$ and an infinite set of polynomials $\mathcal{P}$ for which there is a $K \in J$ satisfying $\| p(T+K)\|_{\mathcal{A}} = \| p(T+K)\|_{\mathcal{A}/J}$ for any finite subset of $\mathcal{P}$ but for which there is no $K \in J$ satisfying $\| p(T+K)\|_{\mathcal{A}} = \| p(T+K)\|_{\mathcal{A}/J}$ for all $p \in \mathcal{P}$. This sheds some light on a well-known question of C. Olsen.

In her work on compact perturbations of operators Olsen [**3**] proposed (and solved in various special cases) the following problem: given $T \in B(H)$ and a polynomial $p(x)$ does there exist a compact operator K so that $\| p(T+K)\| = \| p(T)\|_e$, where $\| \cdot \|_e$ denotes the norm in the Calkin algebra? This question was taken up for a C^*-algebra with a closed ideal by Akemann and Pedersen [**1**] where it was shown that perturbations can be chosen to given equality for the polynomial x^n. A counterexample was given for the polynomial $x^2 - x$ in the algebra $C[0,1]$ of continuous functions on the unit interval. The purpose of this note is to give an example of a C^*-algebra A, an ideal J, an infinite collection of polynomials and an operator $T \in A$ so that a perturbation may be found for any finite number of the polynomials, but no perturbation exists for all of the polynomials simultaneously.

Let S_n be the upper triangular $n \times n$ matrix with 1's on and above the main diagonal. It is easy to see that powers of S_n are also upper triangular and are constant on the diagonals. Let $\alpha_l^{(r)}$ denote the $(1, l)$ entry of $(S_n)^r$. The following lemma will be useful in evaluating $\alpha_l^{(r)}$.

LEMMA 1. *$\sum_{i=1}^m i^k$ is a polynomial $\beta_k(m)$ in m whose leading term is $(k+1)^{-1} m^{k+1}$.*

PROOF. Since $\sum_{i=1}^m i = m(m+1)/2$, the result is true for $k = 1$. Assume the result for $k \leqslant r - 1$ and consider the identity

$$m^{r+1} - 1 = \sum_{i=1}^{m-1} \left[(1+i)^{r+1} - i^{r+1} \right].$$

Expansion of $(1+i)^{r+1}$ combined with the induction hypothesis results in

$$m^{r+1} - 1 = (r+1) \sum_{i=1}^{m-1} i^r + q(m)$$

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where $q(m)$ is a polynomial of degree r. Add $(r+1)m^r$ to both sides to obtain

$$\sum_{i=1}^{m} i^r = (r+1)^{-1}m^{r+1} - (r+1)^{-1}[1 + q(m) - (r+1)m^r],$$

and the induction step is proved.

LEMMA 2. *For $r \geqslant 2$, $\alpha_l^{(r)}$ is obtained by evaluating at $x = l$ a polynomial $p_r(x)$ whose leading term is $x^{r-1}/(r-1)!$.*

PROOF. To begin the induction argument observe that $\alpha_l^{(2)} = l$, $1 \leqslant l \leqslant n$, and so $p_2(x) = x$. Now assume the truth of the lemma for all integers less than or equal to r and write

$$p_r(x) = \sum_{i=0}^{r-1} \lambda_{i,r} x^i$$

where $\lambda_{r-1,r} = 1/(r-1)!$. It follows from the identity $(S_n)^{r+1} = (S_n)^r S_n$ that

$$\begin{aligned}\alpha_q^{(r+1)} &= \sum_{k=1}^{q} \alpha_k^{(r)} = \sum_{k=1}^{q} \sum_{i=0}^{r-1} \lambda_{i,r} k^i \\ &= \sum_{i=0}^{r-1} \left(\lambda_{i,r} \sum_{k=1}^{q} k^i \right) = \sum_{i=0}^{r-1} \lambda_{i,r} \beta_i(q)\end{aligned}$$

by Lemma 1. Then $\alpha_q^{(r+1)}$ is given by evaluation of the polynomial $p_{r+1}(x) = \sum_{i=0}^{r-1} \lambda_{i,r} \beta_i(x)$. The degree of $p_{r+1}(x)$ is r and the leading term is $\lambda_{r-1,r} x^r/r$, which is $x^r/r!$.

LEMMA 3. *For each fixed positive integer $r \geqslant 2$,*

$$\limsup_{n \to \infty} \frac{\|(S_n)^r\|}{n^r} \leqslant \frac{(2r-1)^{-1/2}}{(r-1)!}. \tag{1}$$

PROOF. Let $\{\mathbf{x}_i\}_{i=1}^n$ be the n rows of $(S_n)^r$ and let $\mathbf{y}$ be a unit vector in $\mathbf{C}^n$ at which $(S_n)^r$ attains its norm. The entries of $\mathbf{y}$ may be taken as nonnegative since the entries of $(S_n)^r$ are nonnegative. The ith entry of $(S_n)^r \mathbf{y}$ is $\langle \mathbf{x}_i, \mathbf{y} \rangle$ and so

$$\begin{aligned}\|(S_n)^r\| = \|(S_n)^r \mathbf{y}\| &= \left(\sum_{i=1}^{n} \langle \mathbf{x}_i, \mathbf{y} \rangle^2 \right)^{1/2} \\ &\leqslant \left(\sum_{i=1}^{n} \|\mathbf{x}_i\|^2 \|\mathbf{y}\|^2 \right)^{1/2} = \left(\sum_{i=1}^{n} \|\mathbf{x}_i\|^2 \right)^{1/2} \leqslant n^{1/2} \|\mathbf{x}_1\|\end{aligned}$$

since $\|\mathbf{x}_i\| \leqslant \|\mathbf{x}_1\|$ for $1 \leqslant i \leqslant n$. Consequently

$$\|(S_n)^r\|^2 \leqslant n \sum_{i=1}^{n} \left(\alpha_i^{(r)}\right)^2 = n \sum_{i=1}^{n} \left(p_r(i)\right)^2,$$

and this in turn is a polynomial in n whose leading term is $n^{2r}/(r-1)!^2(2r-1)$ from Lemmas 1 and 2. It follows that (1) holds.

LEMMA 4. *Let T be an operator whose numerical range $W(T)$ contains* 0 *and let D be an operator satisfying* $0 \leqslant D \leqslant I$. *Then, for all* $\lambda \in \mathbf{C}$,

$$\|\lambda + DTD\| \leqslant \|\lambda + T\|.$$

PROOF. Let θ be a state on $B(H)$ satisfying $\theta(T) = 0$, and define a completely positive unital map $\psi\colon B(H) \to B(H)$ by $\psi(X) = \theta(X)(I - D^2) + DXD$. Then $\|\psi\| = 1$ and $\psi(\lambda + T) = \lambda + DTD$. It follows that, for all $\lambda \in \mathbf{C}$, $\|\lambda + DTD\| \leqslant \|\lambda + T\|$.

It is now possible to describe the example mentioned above. Let M_n denote the $n \times n$ matrix algebra and let A be the C^*-algebra $\oplus_{n=1}^{\infty} M_n$, which can be viewed as block-diagonal operators on a separable Hilbert space. Elements of A may be written $T = \oplus_{n=1}^{\infty} T_n$ where it is understood that $T_n \in M_n$. Let J denote the ideal of compact block-diagonal operators. For an operator $T \in A$ its norm in the quotient space will be denoted by $\|T\|_e$, and its image under the quotient map by $\dot{T}$.

Recall the definition of the matrix S_n, and define $T = \oplus_{n=1}^{\infty} S_n/n \in A$. It follows from Lemma 3 that $\|T^r\|_e \leqslant (2r-1)^{-1/2}/(r-1)!$ and so

$$\lim_{r\to\infty} (\|T^r\|_e)^{1/r} \leqslant \lim_{r\to\infty} \frac{(2r-1)^{-1/2r}}{((r-1)!)^{1/r}} = 0.$$

Therefore $\dot{T}$ is quasinilpotent.

Let $P_n \in M_n$ be the projection all of whose entries are $1/n$, and write $P = \oplus_{n=1}^{\infty} P_n \in A$. Then $\dot{P}$ and $\mathrm{Re}(\dot{T})$ are equal and so the numerical range $W(\dot{T})$ of $\dot{T}$ lies in the right half-plane.

THEOREM 1. *Given any finite number*, $q_1(x), \ldots, q_l(x)$, *of polynomials from either of the collections* $\{x^n\colon n \geqslant 1\}$ *or* $\{x + \lambda\colon \lambda \in \mathbf{C}\}$ *there exists an element* $K \in J$ *so that*

$$\|q_i(T)\|_e = \|q_i(T + K)\|, \qquad 1 \leqslant i \leqslant l.$$

However no such K exists for which equality holds simultaneously for all of these polynomials.

PROOF. First choose $K_1 \in J$ so that $\|\lambda + (T + K_1)\| = \|\lambda + T\|_e$ for all $\lambda \in \mathbf{C}$ [**5**, Theorem 1.2, Corollary 1.3]. For a fixed integer n there exists $C \in J$ so that $0 \leqslant C \leqslant I$ and $\|[(T + K_1)(I - C)]^r\| = \|T^r\|_e$ for $1 \leqslant r \leqslant n$ [**1**, Theorem 3.7]. The element C in the proof of this theorem is constructed from any quasicentral approximate identity of J. In this case a diagonal approximate identity may be chosen which is constant on the blocks and so $(I - C)$ commutes with $(T + K_1)$. Write $D = (I - C)^{1/2}$, $0 \leqslant D \leqslant I$, and observe that $D(T + K_1)D$ is a compact perturbation of T by some $K \in J$. Then

$$\|(T + K)^r\| = \|[D(T + K_1)D]^r\| = \|[(T + K_1)(1 - C)]^r\| = \|T^r\|_e$$

for $1 \leqslant r \leqslant n$.

The operator T is essentially quasinilpotent, by construction, and so $0 \in W(T + K_1)$. By Lemma 4

$$\|\lambda + (T + K)\| = \|\lambda + D(T + K_1)D\| \leqslant \|\lambda + (T + K_1)\| = \|\lambda + T\|_e$$

and inequality in the opposite direction is immediate. Thus, for any finite collection of polynomials, the existence of an appropriate $K \in J$ is established.

Suppose, on the other hand, that there exists $K \in J$ so that

$$\|T^r\|_e = \|(T + K)^r\|, \qquad r \geqslant 1,$$

and

$$\|T + \lambda\|_e = \|T + K + \lambda\|, \qquad \lambda \in \mathbf{C}.$$

This would imply that $T + K$ is quasinilpotent and that $W(T + K) = W_e(T)$ [**6**, Theorem 4] and hence lies in the right half-plane. If $T + K$ is written $\bigoplus_{n=1}^{\infty} R_n$, then each R_n is nilpotent and each $W(R_n)$ lies in the right half-plane. Thus, for each n, 0 is on the boundary of the numerical range of the nilpotent matrix R_n. Thus R_n is identically zero [**2**, Theorem 2], which is impossible since $T + K$ is a compact perturbation of the noncompact operator T. This contradiction completes the proof.

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