PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 3, July 1983

THE NUMERICAL RANGE OF A WEIGHTED SHIFT

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ABSTRACT. Let T be a weighted shift on a Hilbert space. We compute the numerical radius of T when T is finite, circular, Hilbert-Schmidt, periodic, or a finite perturbation of periodic. For several cases we also determine whether the numerical range is closed, completing the determination of the numerical range and answering a question of Ridge. An important step is the determination of the eigenvalues of a selfadjoint tri-diagonal matrix with zeroes on its diagonal. We give a simple formula for the eigenvalues when the matrix is finite dimensional or Hilbert-Schmidt.

Let $\mathcal{H}$ be a separable complex Hilbert space. A continuous linear operator T on $\mathcal{H}$ is said to be a (weighted) shift if there is an orthonormal basis $\{f_n\}$ and a bounded sequence of scalars $\{\alpha_n\}$ such that $T(f_n) = \alpha_n f_{n+1}$. Shifts are an important class of operators, and Shields [**11**] provides a good survey of their properties. The *numerical range* of T, denoted $W(T)$, is $\{(Tx, x): \|x\| = 1\}$. The numerical range is always nonempty, convex, and bounded, and for a weighted shift it is circularly symmetric about the origin (see Shields [**11**, Proposition 16]). Because of this symmetry, when T is a weighted shift its numerical range is completely known once one determines

(i) What is the numerical radius of T? (where the numerical radius of T, denoted $w(T)$, is $\max\{|\lambda| : \lambda \in W(T)\}$).

(ii) Is $W(T)$ closed? (always true if T has finite rank).

Shields [**11**, p. 73] has remarked that little is known about these questions. The only relevant papers known to the author are by Ridge [**10**], who showed that the numerical radius of a shift with periodic weights is the same as that of a related circular shift; Berger and Stampfli [**1**, p. 1053], who computed the numerical radius of a unilateral shift with weights $\alpha, 1, 1, \ldots$; and Eckstein [**3**], who showed that if $|a_{2i}|^2 + |a_{2i+1}|^2 \leqslant 2$ for all i then the shift with weights $\{a_i\}$ has a numerical radius no greater than 1.

This paper gives formulae for the numerical radius in several new cases. Our main technique is to analyze $\mathrm{Re}(T)$ ($\mathrm{Re}(T) = (T + T^*)/2$). $\mathrm{Re}(T)$ is selfadjoint, so $w(\mathrm{Re}(T)) = \|\mathrm{Re}(T)\|$. Also, $W(\mathrm{Re}(T))$ is the projection onto the real axis of $W(T)$. If T is a shift then the circular symmetry of $W(T)$ shows that $w(T) = w(\mathrm{Re}(T))$, so one can compute $w(T)$ by computing $\|\mathrm{Re}(T)\|$. If T is of finite rank then so is $\mathrm{Re}(T)$, in which case $\|\mathrm{Re}(T)\|$ equals the maximum of the absolute values of its eigenvalues. Further, $\mathrm{Re}(T)$ is selfadjoint tri-diagonal with zeroes on its diagonal,

Received by the editors September 12, 1980 and, in revised form, August 24, 1981 and August 5, 1982.
1980 *Mathematics Subject Classification*. Primary 47B37, 47A12; Secondary 15A60, 15A18.
[1]Research partially supported by NSF Grant MCS78-02204

and we exploit this form to give a simple polynomial whose roots are the reciprocals of the nonzero eigenvalues of $\mathrm{Re}(T)$. This result is extended to some cases where $\mathrm{Re}(T)$ is infinite-dimensional. Tri-diagonal selfadjoint matrices have been extensively studied (see Wilkinson [**12**, Chapter 5]), but the author has been unable to find previous mention of the formulae derived in this paper.

The author would like to thank the referee for several helpful suggestions.

I. Preliminaries. Let T be a shift where $\{f_n\}$ is an orthonormal basis and $T(f_n) = \alpha_n f_{n+1}$. We say T is *degenerate* if some α_n is zero. A nondegenerate shift is *bilateral* if n ranges through the integers, *unilateral* if n ranges through the positive integers, and *finite* if there is an N such that $n \in \{1,\ldots,N\}$ and $T(f_N) = 0$. A degenerate shift is a direct sum of finite and forward or backward unilateral shifts. The numerical range of a backward shift is that of a forward shift with the weights in reverse order. The numerical range of a direct sum is the convex hull of the numerical ranges of the summands, so it suffices to analyze only nondegenerate shifts. We further assume all weights are positive. This is not really a restriction since a shift with weights $\{\alpha_n\}$ is unitarily equivalent to one with weights $\{|\alpha_n|\}$ (see Shields [**11**, Corollary 11]).

If $\{f_n\}_{n=1}^N$ is an orthonormal basis for $\mathcal{H}$ and if $T(f_n) = \alpha_n f_{n+1}$ for $n < N$ and $T(f_N) = \alpha_N f_1$, then T is called a *circular shift*. Circular shifts are not true shifts, and the unadorned word "shift" does not include them. We denote bilateral, unilateral, finite, and circular shifts by $B(\ldots,\alpha_0,\ldots)$, $U(\alpha_1,\ldots)$, $F(\alpha_1,\ldots,\alpha_n)$, and $C(\alpha_1,\ldots,\alpha_n)$, respectively, where the α's are the weights.

In theory one can compute the numerical radius of an infinite shift by using Lemma 2 below to compute the numerical radius of a finite shift and noting that $w(U(\alpha_1,\alpha_2,\ldots)) = \lim_{n\to\infty} w(F(\alpha_1,\ldots,\alpha_n))$ and $w(B(\ldots,\alpha_0,\ldots)) = \lim_{n\to\infty} w(F(\alpha_{-n},\ldots,\alpha_n))$. However, the limit operation is a formidable obstacle, and it is the purpose of this paper to remove it in certain cases, leaving only a single polynomial to solve (or, for the Schatten classes, an entire function). This reduces an analysis problem to a far simpler algebraic one, permitting exact analysis for many more shifts.

Our formulae involve the circularly symmetric functions $S_r(a_1,\ldots,a_n)$, where n and r are nonnegative integers. S_0 is defined to be 1, while for $r \geqslant 1$, $S_r(a_1,\ldots,a_n) = \Sigma\{\prod_{k=1}^r a_{\pi(k)} \mid \pi\colon (1,\ldots,r) \to (1,\ldots,n)$, where $\pi(k)+1 < \pi(k+1)$ for $1 \leqslant k < r$, and if $\pi(1) = 1$ then $\pi(r) \neq n\}$. These have a nice description: imagine a regular n-gon with vertices labeled a_1 through a_n. Draw a convex r-gon in it, with vertices among the a_i with the restriction that it cannot use an edge of the original polygon. Each term in $S_r(a_1,\ldots,a_n)$ is the product of the vertices of such an r-gon.

These functions satisfy many identities, but we need only the following:

(1) $S_1(a_1,\ldots,a_n) = \Sigma_{k=1}^n a_k$ if $n > 1$.

(2) $S_r(a_1,\ldots,a_n) = 0$ if $r > n/2$.

(3) $S_r(a_1,\ldots,a_n) = S_r(a_2,\ldots,a_n,a_1)$.

(4) $S_r(a_1,\ldots,a_n,0) = S_r(a_1,\ldots,a_n,0,0)$.

(5) $S_{r+1}(a_1,\ldots,a_{n+1},0) = S_{r+1}(a_1,\ldots,a_n,0) + a_{n+1}S_r(a_1,\ldots,a_{n-1},0)$.

II. Finite shifts and extensions.

LEMMA 1. *Let A be an $(n+1)$-dimensional tri-diagonal selfadjoint matrix with zeroes on its diagonal, and let $(a_1,\dots,a_n)$ be the sequence of entries on its subdiagonal. Then*

$$\det(I-\mu A)=\sum_{l=0}^{\lfloor (n+1)/2\rfloor} S_l\big(|a_1|^2,\dots,|a_n|^2,0\big)(-1)^l\mu^{2l}.$$

PROOF. Let A' be the lower right $n\times n$ submatrix of A, and let A'' be the lower right $(n-1)\times(n-1)$ submatrix. Let $p_n(\mu)=\det(I-\mu A)$, $p_{n-1}(\mu)=\det(I-\mu A')$, and $p_{n-2}(\mu)=\det(I-\mu A'')$. Then $p_n(\mu)=p_{n-1}(\mu)-\mu^2|a_1|^2p_{n-2}(\mu)$, and the formula is established by induction. □

Since $\lambda\neq 0$ is an eigenvalue of A if and only if $1/\lambda$ is a root of $\det(I-\mu A)$, we conclude that if A is a finite dimensional tri-diagonal selfadjoint matrix with zeroes on its diagonal then λ is an eigenvalue of A if and only if $-\lambda$ is an eigenvalue. If all subdiagonal entries are nonzero and A is of odd dimension then it has a simple eigenvalue of 0, while if it is of even dimension then 0 is not an eigenvalue. Further, $\|A\|$ is the reciprocal of the smallest positive root of $\det(I-\mu A)$. Recall that if T is a shift then $w(T)=\|\mathrm{Re}(T)\|$, which leads us to the following result.

LEMMA 2. *The numerical radius of the finite weighted shift $F(a_1,\dots,a_n)$ is $1/\sqrt{t}$, where t is the smallest positive root of*

$$0=\sum_{l=0}^{\lfloor (n+1)/2\rfloor}(-1/4)^lS_l\big(a_1^2,\dots,a_n^2,0\big)t^l. \quad \square$$

THEOREM 3. *Let $T=B(\dots,a_0,\dots)$ be Hilbert-Schmidt. Then the numerical radius of T is $1/\sqrt{t}$, where t is the smallest real root of the entire function*

$$\phi(z)=\sum_{l=0}^{\infty}S_l\big(\dots,a_0^2,\dots\big)(-1/4)^lz^l.$$

PROOF. T is Hilbert-Schmidt if and only if $\sum a_k^2<\infty$. To show that ϕ is entire, note that S_1 is defined. Let $\varepsilon(l)=(\sum_{k=-\infty}^{-l}+\sum_{k=l}^{+\infty})a_k^2$. Any term of S_l is a term of S_{l-1} multiplied by an a_k^2 with k in $(-\infty,-l)\cup(l,+\infty)$, and so $S_l\leqslant S_{l-1}\cdot\varepsilon(l)$. Therefore each S_l is defined, and since $\varepsilon(l)\to 0$, ϕ is entire.

To complete the proof, let $F_n=\mathrm{Re}\,F(a_{-n},\dots,a_n)$ and let $q_n(\mu)=\det(I-\mu F_n^2)$. Note that $w(T)=\lim_{n\to\infty}w(F_n)$. The roots of $q_n(\mu)$ are the squares of the roots of $\det(I-\mu F_n)$, and since x is a root of $\det(I-\mu F_n)$ if and only if $-x$ is a root, we conclude that each root of $q_n(\mu)$ is of order two, implying that $q_n(\mu)$ is a square of a polynomial. Using Lemma 2 we see that

$$q_n(\mu)=\left[\sum_{l=0}^{n+1}(-1/4)^l\mu^lS_l\big(a_{-n}^2,\dots,a_n^2,0\big)\right]^2.$$

$\mathrm{Re}(B)$ is Hilbert-Schmidt, so $\mathrm{Re}(B)^2$ is trace class. Thus q_n converges on compact sets to $q(\mu) = \det(I - \mu\,\mathrm{Re}(B)^2)$, and the smallest root of q is $1/\|\mathrm{Re}(B)^2\|$, which equals $1/\|\mathrm{Re}(B)\|^2$. Since $q_n(\mu)$ also converges to $[\sum_{l=0}^{\infty} S_l(\ldots, a_0^2, \ldots)(-1/4)^l \mu^l]^2$ the result is proven. $\square$

Note that the proof proceeds by showing how to compute the nonzero eigenvalues of a Hilbert-Schmidt tri-diagonal selfadjoint operator with zeroes on its diagonal.

Given a weight sequence which is in some l^p, by applying a root-squaring technique we can construct an entire function whose smallest real root is a known power of the numerical radius. For example, suppose the weights $\{a_n\}_{n=-\infty}^{\infty}$ are in l^4. Let $P_n = \det(I - \mu\,\mathrm{Re}[F(a_{-n}, \ldots, a_n)]^4)$ and $P = \det(I - \mu\,\mathrm{Re}[B(\ldots, a_0, \ldots)]^4)$. Then $P_n \to P$ uniformly on compact sets, and also to

$$\left[\sum_{l=0}^{\infty} T_l(\ldots, a_0^2, \ldots)(-1/4)^l \mu^{2l}\right]^2,$$

where

$$T_l(\ldots, a_0^2, \ldots) = \lim_{n\to\infty} \sum_{k=0}^{l} S_k(a_{-n}^2, \ldots, a_n^2, 0) \cdot S_{l-k}(a_{-n}^2, \ldots, a_n^2, 0).$$

Thus $T_0 = 1$, $T_1 = 2\Sigma\, a_n^2$, etc. The numerical radius of $B(\ldots, a_0, \ldots)$ is $s^{-1/4}$, where s is the smallest positive root of $\sum_{l=0}^{\infty} T_l(\ldots, a_0^2, \ldots)(-1/4)^l s^l$.

Unfortunately we see no way of extending Theorem 3 to arbitrary compact shifts. This would require constructing a simple function which is similar to ϕ and is well-defined for arbitrary compact operators.

Despite the concreteness of Theorem 3, it is difficult to apply because of the difficulty in finding roots of entire functions. For example, if the weights are $1, r, r^2, \ldots,\ 0 \leqslant r < 1$, then the corresponding entire function is $\sum_{n=0}^{\infty}[(-1/4)^n z^n r^{2n(n-1)}/\prod_{i=1}^{n}(1 - r^{2i})]$. This is an interesting function, but the author is unable to find its roots.

THEOREM 4. *Let A be a unilateral or bilateral shift with periodic weights $(a_1, \ldots, a_n)$, or let A be $C(a_1, \ldots, a_n)$. Then $w(A)$ is the largest root of*

$$\sum_{l=0}^{\lfloor n/2 \rfloor} S_l(a_1^2, \ldots, a_n^2)(-1/4)^l \lambda^{n-2l} = 2(-1/2)^n \prod_{i=1}^{n} a_i.$$

PROOF. Ridge [**10**, Theorem 1] showed all three shifts have the same numerical radius, but did not determine the polynomial. To do so, let $D = \mathrm{Re}(C(a_1, \ldots, a_n))$, let $q(\lambda) = \det(\lambda I - D)$, and $q_i(\lambda) = \det(\lambda I - \mathrm{Re}\, C(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n))$. By Lemma 2, $q_i = \sum_{l=0}^{\lfloor n/2 \rfloor}(-1/4)^l S_l(a_1^2, \ldots, a_{i-1}^2, 0, a_{i+1}^2, \ldots, a_n^2)\lambda^{n-2l}$. The coefficient of λ^{n-k} in q is a sum of products of k-tuples of distinct entries of D, with no two entries occupying the same row or column. If $k \leqslant n-1$, then each such term occurs in the coefficient of λ^{n-k} for $n-k$ of the q_i's. Summing these shows that when $k \leqslant n-1$, λ^{n-k} has coefficient $S_l(a_1^2, \ldots, a_n^2)(-1/4)^l$ if $k = 2l$, and 0 otherwise.

We now need only determine the coefficient of λ^0. It is a sum of terms of the form $(\prod_{i=1}^n a_i^{m(i)})/2^n$, where $m(i) \in \{0, 1, 2\}$ and $\sum m(i) = n$. If some $m(i)$ is 2 then there is a j such that $m(j)$ is 0, and hence that term is also a term in the coefficient of λ^0 in q_j. Using this we find that the sum of all terms having at least one $m(i) = 2$ is 0 when n is odd and $S_{n/2}(a_1^2, \ldots, a_n^2)$ when n is even. The only terms in the coefficient of λ^0 not yet accounted for are those of the form $(\prod_{i=1}^n a_i)/2^n$. There are 2, both having an odd or even permutation depending on whether n is odd or even. □

For even n one can determine $w(A)$ by solving a polynomial of degree $n/2$, which is smaller than one would initially expect. One reason for interest in bilateral periodic shifts is that they are exactly the reducible nondegenerate bilateral shifts. See Kelley [**7**], Nikolskii [**9**, Theorem 4], or Halmos [**6**, Problem 129].

III. Split periodic shifts. If you take a finite, circular, or Hilbert-Schmidt shift and modify finitely many weights you have one of the same type. However, a finite change of a periodic shift is not so simple. Berger and Stampfli [**1**] analyzed such a shift with weights $\alpha, 1, \ldots$. To handle these perturbations we introduce the class of *split periodic shifts*. We say $A = SP(a_l, \ldots, a_1 \mid b_1, \ldots, b_m \mid c_1, \ldots, c_n)$ if the weights are $\ldots, a_l, a_{l-1}, \ldots, a_1, a_l, \ldots, a_1, b_1, \ldots, b_m, c_1, \ldots, c_n, c_1, \ldots, c_n, \ldots$. This includes more than just finite perturbations of periodic shifts, but the techniques used handle the generality easily. Notice that the similarity class of a bilateral periodic shift is rife with split periodic shifts with different a and c parts and arbitrary nondegenerate b parts (see Shields [**11**, Theorem 2]).

We introduce a new technique to determine the numerical radius, based on a modification of a theorem of Eckstein and Racz [**4**, Theorem 2.5].

THEOREM 5 (ECKSTEIN AND RACZ). *The numerical radius of* $B(\ldots, a_0, \ldots)$, $a_0 \neq 0$, *is* δ, *where* δ *is the smallest positive number for which one can find* $\{e_n\}$ *with* $e_n \in [-1, 1]$ *and* $(a_n/\delta)^2 = (1 + e_{n+1})(1 - e_n)$. □

To illustrate the use of this theorem we calculate $w(SP(0 \mid \alpha \mid \beta, \gamma))$. $SP(0 \mid \alpha \mid \beta, \gamma)$ is just a unilateral shift with weights $\alpha, \beta, \gamma, \beta, \gamma, \ldots$. If α is at the 0th index then

$$e_1 = \left(\alpha^2/\delta^2 - 1 + e_0\right)/\left(1 - e_0\right),$$
$$e_{2i} = \left(\beta^2/\delta^2 - 1 + e_{2i-1}\right)/\left(1 - e_{2i-1}\right), \quad \text{and}$$
$$e_{2i+1} = \left(\gamma^2/\delta^2 - 1 + e_{2i}\right)/\left(1 - e_{2i}\right), \quad i \geqslant 1.$$

For a fixed δ, e_{n+1} is monotone increasing in e_n when e_n is in $(-1, 1)$. Solving for e_{i+2} in terms of e_i, i odd, gives

$$e_{i+2} = \left[(1 - e_i)(\gamma^2/\delta^2 - 2) + \beta^2/\delta^2\right]/\left[2(1 - e_i) - \beta^2/\delta^2\right].$$

The fixed points of this map are

$$\left\{\gamma^2 - \beta^2 \pm \left[\left(\beta^2 - \gamma^2\right)^2 + 8\left(2\delta^4 - \gamma^2\delta^2 - \beta^2\delta^2\right)\right]^{1/2}\right\}/4\delta^2.$$

There is at least one fixed point iff $\delta \geqslant (\gamma + \beta)/2$, i.e., iff $\delta \geqslant w(C(\beta, \gamma))$. Further, both fixed points are in $(-1, 1)$, so the fact that e_{i+2} is an increasing function of e_i

shows that if e_1 is less than the largest fixed point then so are e_3, e_5, etc. The mapping takes the interval from the largest fixed point to 1 onto the interval from the largest fixed point to infinity, so if e_1 is larger than the largest fixed point there will be an odd n such that $e_n > 1$, which is not allowed. Similar reasoning shows that if e_1 is no greater than the largest fixed point then e_2, e_4, etc. will also be in $(-1, 1)$.

The largest fixed point is monotone in δ, so if e_1 were specified it would determine the smallest that δ could be and still be able to find the required $\{e_n\}$. To set the value of e_1, which we want to be as small as possible, we set e_0 to its minimum possible value, -1. Then $e_1 = (\alpha^2/\delta^2 - 2)/2$. Setting this equal to the largest fixed point gives

$$\delta = \alpha \cdot \left[(\alpha^2 + \beta^2 - \gamma^2)/4(\alpha^2 - \gamma^2)\right]^{1/2}.$$

If $\alpha < \gamma$ this is incorrect, for then the numerical radius is $w(C(\beta, \gamma))$. Even if $\alpha > \gamma$ the value may still be incorrect because it may be that $(\alpha^2/\delta^2 - 2)/2$ is less than the largest fixed point for all values of δ greater than or equal to $(\gamma + \beta)/2$, i.e., for all values for which there really is a fixed point. In this case the answer is again $w(C(\beta, \gamma))$. This occurs in the Berger and Stampfli example of $SP(0 \mid \alpha \mid 1)$ for $1 \leqslant \alpha < \sqrt{2}$. In all cases, $w(SP(0 \mid \alpha \mid \beta, \gamma))$ is the maximum of $(\beta + \gamma)/2$ and $\alpha \cdot [(\alpha^2 + \beta^2 - \gamma^2)/(\alpha^2 - \gamma^2)]^{1/2}$.

Technique to compute $w(SP(0 \mid b_1, \ldots, b_m \mid c_1, \ldots, c_n))$. Set $e_0 = -1$ and find e_m in terms of δ. Find e_{m+n} in terms of e_m, find the largest fixed-point of that map, set it equal to e_m, and solve for δ. The numerical radius is the maximum of δ and $w(C(c_1, \ldots, c_n))$.

If the split periodic shift is bilateral then it is not true that $e_0 = -1$. As an example, consider $SP(\alpha \mid \beta \mid \gamma)$. Now $e_0 = ((\alpha/\delta)^2 - 1 + e_{-1})/(1 - e_{-1})$. Solving $x = ((\alpha/\delta)^2 - 1 + x)/(1 - x)$ we find $x^2 = 1 - (\alpha/\delta)^2$. The monotonicity of e_{n+1} in terms of e_n shows that e_0 is at least $-[1 - (\alpha/\delta)^2]^{1/2}$, and similarly e_1 is at most $[1 - (\gamma/\delta)^2]^{1/2}$. We now solve the equations

$$e_1 = \sqrt{1 - (\gamma/\delta)^2},$$
$$e_1 = \left[(\beta/\delta)^2 - 1 - \sqrt{1 - (\alpha/\delta)^2}\right] \Big/ \left[1 + \sqrt{1 - (\alpha/\delta)^2}\right]$$

for δ, obtaining a fourth degree polynomial in δ^2 which simplifies for certain cases. For example, when $\alpha = \gamma$, $\delta = (\alpha^2 + \beta^2)/2\beta$. This is the numerical radius if $\beta \geqslant \alpha$, while if $\beta < \alpha$ the radius is α. To compute $w(SP(a_l, \ldots, a_1 \mid b_1, \ldots, b_m \mid c_1, \ldots, c_n))$, in general, first find δ, and then the answer is the maximum of δ, $w(C(a_1, \ldots, a_l))$, and $w(C(c_1, \ldots, c_n))$.

The techniques of this section can be used to find the norm of those tri-diagonal selfadjoint matrices which have zeroes on their diagonal and a split-periodic subdiagonal. Unfortunately, the complexity of the computations increases rapidly as the number of parameters increases.

IV. Closure of the numerical range. For the cases considered above we determine whether the numerical range is a closed or open disk. First we establish a conjecture of Ridge [**10**, Note 5].

PROPOSITION 6. *Let A be a unilateral or bilateral shift with periodic weights. Then $W(A)$ is an open disk.*

PROOF. Our proof is just a reworking of the Perron-Frobenius theorem. (See Gantmacher [**5**, p. 53].) It suffices to consider only bilateral periodic shifts, so let A be such a shift and let $r = w(A)$. $W(A)$ is closed if and only if there is a unit vector $x = (x_i)$ such that $\langle Ax, x\rangle = r$. $\langle \mathrm{Re}(A)x, x\rangle = r$, and since $\|\mathrm{Re}(A)\| = r$, $\mathrm{Re}(A)x = rx$. If p is the period of A's weights then also $\mathrm{Re}(A)(x_{i+p}) = r(x_{i+p})$. Letting $y^* = (x_i - x_{i+p})$ and $y = y^*/\|y^*\|$, we also have $\mathrm{Re}(A)y = ry$ and $\langle Ay, y\rangle = r$. Since $\|\mathrm{Re}\, A(|x_i|)\| \geqslant \|\mathrm{Re}(A)x\| = \|\mathrm{Re}(A)\|$, we may assume $x_n \geqslant 0$ for all n. This assumption then forces y to have some components positive and some negative, which we will show is impossible.

Let $y = (y_i)$ and assume there is a k such that $y_k y_{k+1} < 0$. Then $\langle A(|y_i|), (|y_i|)\rangle > \langle Ay, y\rangle = w(A)$, which is impossible. Therefore it must be that between components of y of opposite sign there is a 0 component. Assume k is such that $y_k \neq 0$ and $y_{k+1} = 0$. Define the vector $z(\delta) = (z_i(\delta))$ by $z_i(\delta) = |y_i|$, $i \neq k+1$, and $z_{k+1}(\delta) = \delta$. $\langle Az(0), z(0)\rangle = r$ and $\langle Az(\delta), z(\delta)\rangle$ increases linearly with δ, while $\|z(\delta)\| = 1 + \delta^2$. For sufficiently small δ, $\langle Az(\delta), z(\delta)\rangle/\|z(\delta)\|^2 > \langle Ay, y\rangle = w(A)$, which again is impossible. Therefore x cannot exist, and $W(A)$ is open. □

Let A be a bounded linear operator on $\mathcal{H}$. The *essential numerical range* of A, denoted $W_e(A)$, is given by $W_e(A) = \bigcap\{\overline{W(A+K)}: K \text{ a compact operator on } \mathcal{H}\}$. $W_e(A)$ is always compact, convex, nonempty, and contained in $\overline{W(A)}$. A is compact iff $W_e(A) = \{0\}$. (See Bonsall and Duncan [**2**, §34].)

PROPOSITION 7. *If A is a weighted shift and $W_e(A) \neq \overline{W(A)}$ then $W(A)$ is a closed disk.*

PROOF. Lancaster [**8**] showed that the extreme points of $\overline{W(A)}$ lie in $W_e(A) \cup W(A)$. □

COROLLARY 8. *Any compact weighted shift has a closed numerical range.* □

COROLLARY 9. *Let $A = SP(a_l, \ldots, a_1 \mid b_1, \ldots, b_m \mid c_1, \ldots, c_n)$. If*

$$w(A) > \max\{w(C(a_1, \ldots, a)), w(C(c_1, \ldots, c_n))\}$$

then $W(A)$ is a closed disk.

PROOF.

$$\begin{aligned} W_e(A) &= W_e(SP(a_l, \ldots, a_1 \mid 0 \mid c_1, \ldots, c_n)) \\ &= W(C(a_l, \ldots, a_1)) \cup W(C(c_1, \ldots, c_m)). \qquad \square \end{aligned}$$

V. Notes. (1) It would be useful to have formulae for the numerical radius of, say, a shift which is a Hilbert-Schmidt perturbation of $U(1, 1, \ldots)$. Using techniques from §III one can show that if $a_i \geqslant 1$ and $\sum i(a_i^2 - 1) > 1$, then $w(U(a_1, \ldots)) > 1$.

(2) The algebraic form of our results suggests several questions. For example, the polynomial in Lemma 1 has a special form. What polynomials arise in this manner?

(3) We have not determined the closure of $W(A)$ when

$$A = SP(a_l \cdots a_1 \mid b_1 \cdots b_m \mid c_1 \cdots c_n)$$

and

$$w(A) = \max\{w(F(a_1 \cdots a_l)), w(F(c_1 \cdots c_n))\},$$

but we conjecture that in this case $W(A)$ is open.

(4) From all published examples one might conjecture that if A is a shift for which $W_e(A) = \overline{W(A)}$, then $W(A)$ is open. However, easy perturbation arguments show that there are nondegenerate shifts which are trace-class perturbations of $U(2, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots)$ and which have equal numerical range and essential numerical range.

References

1. C. A. Berger and J. G. Stampfli, *Mapping theorems for the numerical range*, Amer. J. Math. **89** (1967), 1047–1055.

2. F. F. Bonsall and J. Duncan, *Numerical ranges*. II, London Math. Soc. Lecture Note Series, vol. 10, Cambridge Univ. Press, London, 1973.

3. G. Eckstein, *Un exemple d'un opérateur de class $\mathcal{C}_\rho$ a fonction charactéristique non bornée*, C. R. Acad. Sci. Paris Sér. A **274** (1972), 621–623.

4. G. Eckstein and A. Racz, *Weighted shifts of class $\mathcal{C}_\rho$*, Acta Sci. Math. **35** (1973), 187–194.

5. R. F. Gantmacher, *The theory of matrices*. Vol. 2, Chelsea, New York, 1960.

6. P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967.

7. R. L. Kelley, *Weighted shifts on Hilbert space*, Dissertation, Univ. of Michigan, Ann Arbor, Mich., 1966.

8. J. Lancaster, *The boundary of the numerical range*, Proc. Amer. Math. Soc. **49** (1975), 393–398.

9. N. K. Nikolskii, *On invariant subspaces of weighted shift operators*, Mat. Sb. **74** (116) (1967), 171–190 (Russian); transl. in Math. USSR-Sb. **3** (1967), 159–176.

10. W. C. Ridge, *Numerical range of a weighted shift with periodic weights*, Proc. Amer. Math. Soc. **55** (1976), 107–110.

11. A. L. Shields, *Weighted shift operators and analytic function theory*, in Topics in Operator Theory (C. Pearcy, Editor), Math. Surveys, vol. 13, Amer. Math. Soc., Providence, R. I., 1974.

12. J. H. Wilkinson, *The algebraic eigenvalue problem*, Oxford Univ. Press, London, 1965.

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