EXISTENCE OF HARMONIC $L^1$ FUNCTIONS IN COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. We construct a complete Riemannian manifold which carries a nonconstant harmonic $L^1$ function.

First we consider the dimension 2 case. Let the manifold be $Z = \mathbb{R} \times S$ with parameterization $(r, \theta)$, $-\infty < r < \infty$ and $0 \leq \theta \leq 2\pi$, with identification of 0 and $2\pi$ for $\theta$. On $Z$, endow the Riemannian metric $ds^2 = f(r)(dr^2 + d\theta^2)$ where $f$ is a positive $C^\infty$ function such that $f(r) = (r \log |r|)^{-2}$ for $|r| > 2$. Clearly the metric is complete, since

$$
\int_2^\infty f(r) \, dr = \int_2^\infty (r \log r)^{-1} \, dr = \infty = \int_{-\infty}^{-2} f(r) \, dr.
$$

On the other hand, the function $h(r, \theta) = r$ is harmonic since

$$
\Delta r = f(r)^{-1} \frac{\partial^2}{\partial r^2} r = 0.
$$

It is also $L^1$, since

$$
\int_{-\infty}^{\infty} f(r) \cdot |r| \, dr = \int_{-\infty}^{-2} + \int_{-2}^{2} + \int_{2}^{\infty} f(r) \cdot |r| \, dr < \infty
$$

by the simple observation that $\int_{-\infty}^{\infty} (r \log r)^{-2} r \, dr < \infty$.

For dimension $N > 2$, let $T^{N-2}$ be the $N-2$ torus with a flat metric. Form the product manifold $Z \times T^{N-2}$ with the product metric. Then exactly the same argument as before goes.

Remark and acknowledgement. The above result was obtained quite a while ago and was announced in [1, p. 70]. In view of the result of Yau [2] that for $p \neq 1$, no complete Riemannian manifolds carry such a harmonic $L^p$ function, we feel that this complements his result very nicely and may probably have some interest.

Our result was quoted recently in p. 135 of [3]. However, there is a misprint there. In [3] the volume of our example was asserted to be infinite; actually it is finite. We do not know what happens when the volume is infinite.

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REFERENCES


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