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EXISTENCE OF HARMONIC L^1 FUNCTIONS IN COMPLETE RIEMANNIAN MANIFOLDS

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ABSTRACT. We construct a complete Riemannian manifold which carries a nonconstant harmonic L^1 function.

First we consider the dimension 2 case. Let the manifold be $Z = \mathbf{R} \times S$ with parameterization (r, θ), $-\infty < r < \infty$ and $0 \leqslant \theta \leqslant 2\pi$, with identification of 0 and 2π for θ. On Z, endow the Riemannian metric $ds^2 = f(r)[dr^2 + d\theta^2]$ where f is a positive C^∞ function such that $f(r) = (r \log |r|)^{-2}$ for $|r| \geqslant 2$. Clearly the metric is complete, since

$$\int_2^\infty \sqrt{f(r)}\, dr = \int_2^\infty (r \log r)^{-1}\, dr = \infty = \int_{-\infty}^{-2} \sqrt{f(r)}\, dr.$$

On the other hand, the function $h(r, \theta) = r$ is harmonic since

$$\Delta r = f(r)^{-1} \frac{\partial^2}{\partial r^2} r = 0.$$

It is also L^1, since

$$\int_{-\infty}^{\infty} f(r) \cdot |r|\, dr = \int_{-\infty}^{-2} + \int_{-2}^{2} + \int_2^\infty f(r)\, |r|\, dr < \infty$$

by the simple observation that $\int_2^\infty (r \log r)^{-2} r\, dr < \infty$.

For dimension $N > 2$, let T^{N-2} be the $N-2$ torus with a flat metric. Form the product manifold $Z \times T^{N-2}$ with the product metric. Then exactly the same argument as before goes.

REMARK AND ACKNOWLEDGEMENT. The above result was obtained quite a while ago and was announced in [**1**, p. 70]. In view of the result of Yau [**2**] that for $p \neq 1$, no complete Riemannian manifolds carry such a harmonic L^p function, we feel that this complements his result very nicely and may probably have some interest.

Our result was quoted recently in p. 135 of [**3**]. However, there is a misprint there. In [**3**] the volume of our example was asserted to be infinite; actually it is finite. We do not know what happens when the volume is infinite.

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