STRONG CONVERGENCE OF MARTINGALES
IN VON NEUMANN ALGEBRAS

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Abstract. In this paper we prove strong and $L^1$-norm convergence of martingales with respect to a faithful normal semifinite weight on a von Neumann algebra.

1. Introduction. Let $M$ be a von Neumann algebra and $\varphi$ be a faithful normal semifinite weight on $M_+$ (the positive cone of $M$). We denote by $n_\varphi$ the set of all elements $x \in M$ with $\varphi(x^*x) < +\infty$ and by $m_\varphi$ the linear span of $n_\varphi^* n_\varphi$. Then $\varphi$ is uniquely extended to a linear functional on $m_\varphi$ and we also denote it by $\varphi$. For a von Neumann subalgebra $N$ of $M$, if there exists a faithful $\sigma$-weakly continuous projection $\varepsilon$ of norm one from $M$ onto $N$ such that $\varphi(x) = \varphi(\varepsilon(x))$ for every $x \in m_\varphi$, we call $\varepsilon$ the conditional expectation onto $N$ with respect to $\varphi$. Takesaki [6] proved that there exists a unique conditional expectation onto $N$ with respect to $\varphi$ if and only if $\varphi$ is semifinite on $N$ (i.e., $N \cap m_\varphi$ is $\sigma$-weakly dense in $N$) and $N$ is globally invariant under the modular automorphism group associated with $\varphi$.

Let $M_*$ be the predual of $M$ (i.e., the set of all $\sigma$-weakly continuous linear functionals on $M$). Suppose that $N$ is a von Neumann subalgebra of $M$ such that the conditional expectation $\varepsilon$ onto $N$ with respect to $\varphi$ exists. Then we define $L^1(N; \varepsilon)$ as the set of all elements $\psi \in M_*$ with $\psi = \psi \circ \varepsilon$. For any $\psi \in M_*$ we define $\varepsilon^*(\psi) = \psi \circ \varepsilon$.

Theorem 1. $L^1(N; \varepsilon)$ is isometrically isomorphic to the predual of $N$. Moreover $\varepsilon^*$ is a projection of norm one from $M_*$ onto $L^1(N; \varepsilon)$.

Proof. Let $N_*$ be the predual of $N$. For any $\psi \in N_*$ we define $\iota(\psi) = \psi \circ \varepsilon$. Then $\iota(\psi) \in L^1(N; \varepsilon)$ and

$$\|\psi\| = \sup_{\|x\| \leq 1} |\psi(x)| = \sup_{\|x\| \leq 1} |\psi(\varepsilon(x))| = \|\iota(\psi)\|,$$

because $\varepsilon$ is a projection of norm one from $M$ onto $N$. Moreover $\iota(\psi \uparrow N) = \psi$ for every $\psi \in L^1(N; \varepsilon)$. Hence $\iota$ is an isometrical isomorphism from $N_*$ onto $L^1(N; \varepsilon)$. On the other hand for any $\psi \in M_*$ it follows that

$$\|\varepsilon^*(\psi)\| = \sup_{\|x\| \leq 1} |\psi(\varepsilon(x))| \leq \sup_{\|x\| \leq 1} |\psi(x)| = \|\psi\|.$$

Thus $\varepsilon^*$ is of norm one. It is clear that $\varepsilon^*$ is a projection from $M_*$ onto $L^1(N; \varepsilon)$.

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Suppose that \( \varphi \) is a state. Goldstein [4] embedded \( M \) into \( M_* \) and defined \( L^1(N) \) as the closure of \( N \) in \( M_* \) for any von Neumann subalgebra \( N \) of \( M \). If the conditional expectation \( e \) onto \( N \) with respect to \( \varphi \) exists, \( L^1(N) \) coincides with \( L^1(N; e) \) defined as above. In that paper he proved the strong convergence of martingales. We generalize this to the case in which \( \varphi \) is a weight.

2. Increasing martingales. In this section we fix an increasing net \( \{N_a\} \) of von Neumann subalgebras of \( M \) such that the conditional expectation \( e_a \) onto \( N_a \) with respect to \( \varphi \) exists for every \( a \).

**Theorem 2.** The conditional expectation \( \epsilon_\infty \) onto \( \bigvee_{a} N_a \) with respect to \( \varphi \) exists and the following assertions are satisfied:

(i) \( \{\epsilon_{a}(x)\} \) converges to \( \epsilon_\infty(x) \) in the \( s^* \)-topology for every \( x \in M \);

(ii) \( \{\epsilon_\infty^*(\psi)\} \) converges to \( \epsilon_\infty^*(\psi) \) in the norm for every \( \psi \in M_*^\text{s} \).

**Proof.** Since \( \varphi \) is semifinite on \( N_a \) and \( N_a \) is globally invariant under the modular automorphism group associated with \( \varphi \) for every \( a \), \( \varphi \) is semifinite on \( \bigvee_{a} N_a \) and \( \bigvee_{a} N_a \) is globally invariant under the modular automorphism group. Hence there exists a unique conditional expectation \( \epsilon_\infty \) onto \( \bigvee_{a} N_a \) with respect to \( \varphi \). By the uniqueness of the conditional expectation \( \epsilon_a = \epsilon_\infty \circ \epsilon_\infty \) for every \( a \). Therefore to prove (i) and (ii) we may assume that \( M = \bigvee_{a} N_a \) without loss of generality.

(i) We first prove that if \( \epsilon_a(x) = 0 \) for every \( a \), then \( x = 0 \). Assume that \( \epsilon_\infty(x) = 0 \) for every \( \epsilon_\infty(x) = 0 \) for every \( a \). We fix an index \( a_0 \) and for any \( \psi \in M_* \) we put \( \hat{\psi} = \psi \circ \epsilon_{a_0} \). Then for any \( \alpha \geq a_0 \) and \( a \in N_a \) we have \( \hat{\psi}(ax) = \hat{\psi}(\epsilon_{a_0}(ax)) = \hat{\psi}(ae_{a_0}(x)) = 0 \). Therefore \( \psi(ax) = 0 \) for any \( a \in \bigcup_{a} N_a \). Since \( \bigcup_{a} N_a \) is \( \sigma \)-weakly dense in \( M \), we have \( \hat{\psi}(ax) = 0 \) for any \( a \in M \). Hence \( \psi(\epsilon_{a_0}(x*x)) = \hat{\psi}(x*x) = 0 \). Since \( \psi \in M_* \) is arbitrary, \( \epsilon_{a_0}(x*x) = 0 \), and since \( \epsilon_{a_0} \) is faithful, it follows that \( x = 0 \). Next we prove that \( \{\epsilon_{a}(x)\} \) converges to \( x \) in the \( \sigma \)-weak topology for any \( x \in M \). Since \( \{\epsilon_{a}(x)\} \) is uniformly bounded, for any subnet \( \{\epsilon_{a}(x)\} \) there exists a subnet \( \{\epsilon_{a}(x)\} \) which converges to some \( y \in M \) in the \( \sigma \)-weak topology. For every \( \alpha \), because of \( \sigma \)-weak continuity of \( \epsilon_{a} \), it follows that \( \epsilon_{a}(\epsilon_{a}(x)) \) tends to \( \epsilon_{a}(y) \) as \( \alpha \uparrow \) in the \( \sigma \)-weak topology. Here for sufficiently large \( \alpha \) we have \( \epsilon_{a}(\epsilon_{a}(x)) = \epsilon_{a}(x) \). Hence \( \epsilon_{a}(x) = \epsilon_{a}(y) \) for every \( \alpha \) and we have \( x = y \) by the fact that is proved above. Thus \( \{\epsilon_{a}(x)\} \) converges to \( x \) in the \( \sigma \)-weak topology. Moreover for any \( \psi \in M_*^\text{s} \) it follows that

\[
0 \leq \psi((\epsilon_{a}(x) - x)^*(\epsilon_{a}(x) - x)) = \psi(\epsilon_{a}(x)^*\epsilon_{a}(x)) - \psi(\epsilon_{a}(x)^*x) - \psi(x^*\epsilon_{a}(x)) + \psi(x)*x) \\
\leq \psi(\epsilon_{a}(x)^*x) - \psi(\epsilon_{a}(x)^*x) - \psi(x^*\epsilon_{a}(x)) + \psi(x)*x) \\
\rightarrow 0 \text{ (as } \alpha \uparrow \text{).}
\]

\( \psi((\epsilon_{a}(x) - x)(\epsilon_{a}(x) - x)^*) \) tends to 0 as \( \alpha \uparrow \) in the same way and we have (i).

(ii) Since \( \{\epsilon_{a}(x)\} \) converges to \( x \) in the \( \sigma \)-weak topology for any \( x \in M \), \( \{\epsilon_{a}(\psi)\} \) converges to \( \psi \) in the weak topology for any \( \psi \in M_* \). Therefore \( \bigcup_{a} L^1(N_a; e_a) \) is weakly dense in \( M_* \) and so in the norm. Hence we have (ii) by the standard argument.
**Remark.** The strong convergence of martingales was proved by Connes [2] for the case in which \( M \) is \( \sigma \)-finite, and by Lance [5] for the case in which \( \bigvee_{\alpha} N_\alpha \) is semifinite.

3. Decreasing martingales. In this section we fix a decreasing net \( \{ N_\alpha \} \) of von Neumann subalgebras of \( M \) such that the conditional expectation \( \epsilon_\alpha \) onto \( N_\alpha \) with respect to \( \varphi \) exists for every \( \alpha \).

**Theorem 3.** Suppose that \( \varphi \) is semifinite on \( \bigcap_\alpha N_\alpha \). Then the conditional expectation \( \epsilon_\infty \) onto \( \bigcap_\alpha N_\alpha \) with respect to \( \varphi \) exists and the following assertions are satisfied:

(i) \( \{ \epsilon_\alpha(x) \} \) converges to \( \epsilon_\infty(x) \) in the \( \sigma^* \)-topology for every \( x \in M \);

(ii) \( \{ \epsilon^*_\alpha(\psi) \} \) converges to \( \epsilon^*_\infty(\psi) \) in the norm for every \( \psi \in M^*_\ast \).

**Proof.** Since \( N_\lambda \) is globally invariant under the modular automorphism group associated with \( \varphi \) for every \( \alpha \), so is \( \bigcap_\alpha N_\alpha \), and since \( \varphi \) is semifinite on \( \bigcap_\alpha N_\alpha \), there exists a unique conditional expectation \( \epsilon_\infty \) onto \( \bigcap_\alpha N_\alpha \) with respect to \( \varphi \). By the uniqueness of the conditional expectation, \( \epsilon_\infty = \epsilon_\infty \circ \epsilon_\alpha \) for every \( \alpha \).

(ii) Let any \( \psi \in M^*_\ast \) be fixed. Then

\[
\| \epsilon^*_\alpha(\psi) - \epsilon^*_\infty(\psi) \| = \sup_{\| x \| \leq 1} | \psi(\epsilon_\alpha(x)) - \psi(\epsilon_\infty(x)) |
\]

for some \( x_\alpha \in N_\alpha \) with \( \| x_\alpha \| \leq 1 \). Since \( \{ x_\alpha \} \) is uniformly bounded, for any subnet \( \{ x_{\alpha'} \} \) there exists a subnet \( \{ x_{\alpha^{''}} \} \) which converges to some \( x_\infty \in M \) in the \( \sigma \)-weak topology. Then \( x_\infty \in \bigcap_\alpha N_\alpha \) and \( | \psi(x_{\alpha^{''}}) - \psi(\epsilon_\infty(x_{\alpha^{''}})) | \) tends to 0 as \( \alpha^{''} \uparrow \). Therefore it follows that \( \{ \epsilon^*_\alpha(\psi) \} \) converges to \( \epsilon^*_\infty(\psi) \) in the norm.

(i) It can be easily seen that \( \{ \epsilon_\alpha(x) \} \) converges to \( \epsilon_\infty(x) \) in the \( \sigma \)-weak topology in the same way as the proof of Theorem 2(i). For any \( \psi \in M^*_\ast \) we have

\[
\psi((\epsilon_\alpha(x) - \epsilon_\infty(x))(*(\epsilon_\alpha(x) - \epsilon_\infty(x)))) = \psi(\epsilon_\alpha(x) * \epsilon_\alpha(x)) - \psi(\epsilon_\alpha(x) * \epsilon_\infty(x))
\]

Here the second and third terms of the right-hand side tend to \( \psi(\epsilon_\infty(x) * \epsilon_\infty(x)) \) as \( \alpha \uparrow \). Moreover the first term is equal to \( \epsilon^*_\alpha(\psi)(\epsilon_\alpha(x) * x) \). Here \( \epsilon^*_\alpha(\psi) \) tends to \( \epsilon^*_\infty(\psi) \) as \( \alpha \uparrow \) in the norm and \( \{ \epsilon_\alpha(x) \} \) is uniformly bounded and converges to \( \epsilon_\infty(x) \) in \( \sigma \)-weak topology. So it follows that \( \epsilon^*_\alpha(\psi)(\epsilon_\alpha(x) * x) \) tends to \( \epsilon^*_\infty(\psi)(\epsilon_\infty(x) * x) = \psi(\epsilon_\infty(x) * \epsilon_\infty(x)) \) as \( \alpha \uparrow \). Thus \( \psi((\epsilon_\alpha(x) - \epsilon_\infty(x))(*(\epsilon_\alpha(x) - \epsilon_\infty(x)))) \) tends to 0 as \( \alpha \uparrow \).

In the above theorem the condition that \( \varphi \) is semifinite on \( \bigcap_\alpha N_\alpha \) is necessary. Let \( M = L^\infty \) and for \( \{ x_i \} \subseteq M^+ \) define \( \varphi((x_i)) = \Sigma_{i=1}^\infty x_i \). Then \( M \) is a von Neumann algebra and \( \varphi \) is a faithful normal semifinite weight on \( M^+ \). Now let \( N_\lambda \) be the set of all elements \( \{ x_i \} \subseteq M \) such that \( x_1 = x_2 = \cdots = x_n \). Then \( \{ N_\lambda \} \) is a decreasing sequence of von Neumann subalgebras of \( M \). We can easily see that the conditional
expectation $\varepsilon_n$ onto $N_n$ with respect to $\varphi$ exists and for any $\{x_i\} \in M$, $\varepsilon_n(\{x_i\})$ is the sequence $\{y_i\}$ such that $y_i = (1/n) \cdot \sum_{k=1}^{n} x_k$ for $i = 1, \ldots, n$ and $y_i = x_i$ for $i = n + 1, \ldots$. Then $N_n = \{\lambda \cdot I : \lambda \in \mathbb{C}\}$, on which $\varphi$ is not semifinite. Moreover $\{\varepsilon_n(\{x_i\})\}$ does not always converge in the $\sigma$-weak topology, because $\{(1/n) \cdot \sum_{k=1}^{n} x_k\}$ does not always converge.

In the increasing case (resp. the decreasing case) a sequence $\{x_\alpha\}$ is called a martingale if $x_\alpha \in N_\alpha$ for every $\alpha$ and $\varepsilon_\alpha(x_\beta) = x_\alpha$ whenever $\alpha \leq \beta$ (resp. $\alpha \geq \beta$). The sequence $\{\varepsilon_\alpha(x)\}$ in Theorem 2 (resp. Theorem 3) is an example of martingales and such a martingale is called simple. In the decreasing case any martingale $\{x_\alpha\}$ is essentially simple by considering $\{x_\alpha\}_{\alpha \geq \alpha_0}$ for any fixed $\alpha_0$. In the increasing case, examining the proof of Theorem 2, we can easily see that a martingale is simple if and only if it is uniformly bounded. Similarly, $L^1$-martingales in $M_*$ can be considered and it is seen that an increasing $L^1$-martingale is simple if it is relatively weakly compact in $M_*$. 


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References


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