MINIMAL IDEALS IN QUADRATIC JORDAN ALGEBRAS

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Abstract. In associative and alternative algebras a minimal ideal is either trivial or simple. This is not known for quadratic Jordan algebras. In the present note we show that a minimal ideal is either trivial or \( \mathcal{D} \)-simple (possesses no proper ideals invariant under all inner derivations induced from the ambient algebra). In particular, the heart of any quadratic Jordan algebra is either trivial or \( \mathcal{D} \)-simple. Hearts have recently played an important role in Zelmanov's theory of prime Jordan algebras.

A unital quadratic Jordan algebra over an arbitrary ring of scalars \( \Phi \) is a \( \Phi \)-module \( J \) with unit element 1 and composition \( U_xy \) quadratic in \( x \) and linear in \( y \) such that the axioms

\[
\begin{align*}
(0.1) & \quad U_1 = 1, \\
(0.2) & \quad U_x V_{x,x} = V_{x,x} U_x = U_{U(x)y,x}, \\
(0.3) & \quad U_{U(x)y} = U_x U_y
\end{align*}
\]

hold strictly, where we introduce the notations \( V_{x,y}(z) = \{ x y z \} = U_{x,z}(y) \) for \( U_{x,z} = U_{x,z} - U_x - U_z \); the square is given by \( x^2 = U_x1 \), with linearization \( V_x(y) = x \circ y = U_{x,y}1 \). A (nonunital) quadratic Jordan algebra \( J \) has compositions \( U_{x,y} \) and \( x^2 \) satisfying certain axioms which guarantee that its unital hull \( \bar{J} = \Phi 1 + J \) is a unital quadratic Jordan algebra.

A derivation of a quadratic Jordan algebra is a linear transformation \( D \) such that

\[
D(x^2) = x \circ D(x), \quad D(U_{x,y}) = U_{D(x),x} y + U_x D(y).
\]

The inner derivations are given by

\[
D_{x,y} = [V_x, V_y] = V_{x,y} - V_{y,x}.
\]

An ideal \( I \lhd J \) is a subspace which is both an outer ideal (\( U_I J \subset I \), i.e. \( U_I J \subset I \) and \( V_I J \subset I \)) and an inner ideal (\( U_I \bar{J} \subset I \), i.e. \( U_I \bar{J} \subset I \) and \( I^2 \subset I \)). An outer ideal \( I \) is inner as soon as \( U_I \bar{J} \subset I \) for some spanning set \( \{ i \} \) for \( I \). An algebra is simple if it is not trivial and has no proper ideals; it is \( \mathcal{D} \)-simple if it is not trivial and has no proper ideals which are invariant under all derivations.

A subideal \( K \) of \( J \) is a subspace which is \( n \) steps removed (for some \( n \)) from being an ideal, \( K \lhd I_1 \lhd I_2 \lhd \cdots \lhd I_n \lhd J \). We are interested in how far a subideal is
from being an ideal. The question of whether a minimal ideal \( I \triangleleft J \) is simple (resp. \( \rho \)-simple) depends on whether there are any proper (resp. \( \rho \)-invariant) subideals \( K \triangleleft I \triangleleft J \); our goal will be to show that each such \( K \) would be an ideal in \( J \), therefore by minimality of \( I \) either \( K = I \) or \( K = 0 \), so \( K \) would be improper.

In general, if \( I, K \triangleleft J \) then their quadratic product \( U_IK \triangleleft J \) also. In particular, if \( I \triangleleft J \) then \( I^3 = U_II \triangleleft J \) also. We say \( I \) is trivial if \( I^3 = 0 \) (this is weaker than requiring \( I^2 = 0 \)) and idempotent if \( I^3 = I \). If \( I \) is a minimal ideal in \( J \), then the \( J \)-ideal \( I^3 \subset I \) can only be 0 or \( I \), so \( I \) is either trivial or idempotent.

1. Subideals. We gather here some technical results on subideals of idempotent ideals. These require some further properties of multiplication operators in Jordan algebras (see [2]):

\[
\begin{align*}
(1.1) & \quad V_x = U_{x^1} = V_{x,1} = V_{1,x}, \\
(1.2) & \quad \{xxz\} = x^2 \circ z, \quad \{xyz\} + \{yxz\} = (x \circ y) \circ z, \\
(1.3) & \quad V_{zz} = V_{z^2}, \quad V_{zy} = V_{z,y} - V_{z,y}, \quad 2U_z = V_z^2 - V_{zz}, \\
(1.4) & \quad V_{U(x)y,z} = V_{x, U(y)z}, \\
(1.5) & \quad V_{U(x)y,z} = V_{x, y} - U_xU_yU_z, \quad V_{U(x)y} = V_{x,y} - U_{x,y}U_x, \\
(1.6) & \quad V_{U(x)y} = V_{x,y} - U_{x,y}V_x = V_{x,y} - U_{x,y}U_x, \\
(1.7) & \quad V_{x,y}U_z + U_zV_{x,y} = U_{(xyz)}U_z, \quad VU_z + U_zV_x = V_{x^*}U_z, \\
(1.8) & \quad U_xU_yU_z + U_zU_xU_y + V_{x,y}U_{z,y} = U_{(xyz)}U_z + U_{(xyz)}U_z.
\end{align*}
\]

We always have an expression for the \( I \)-ideal generated by \( K \).

1.9 Lemma. The ideal in \( J \) generated by a subspace \( K \) is the outer hull \( \overline{K} = \sum_{n=0}^{\infty} U_I^nL \) of \( L = K + UKJ \). If \( K \triangleleft I \triangleleft J \), then \( L \triangleleft I \) with \( U_I\overline{K} \subset K \).

Proof. Clearly \( K \) and \( L \) generate the same ideal in \( J \) since this must contain \( U_K\overline{I} \) by \( I \)-innerness. Clearly \( \sum U_I^nL \) is an outer ideal in \( J \) containing \( L \), and it is an inner ideal, since for the spanning elements \( t = U_{a_1} \cdots U_{a_s} (s = k \text{ or } U_k a. k \in K, a_i \in \overline{I}) \) we have \( U_I\overline{I} = U_{a_1} \cdots U_{a_s}U_{a_1} \cdots U_{a_s} \) (by (0.3)) \( \subset U_I^nU_I\overline{I} \) (since \( U_I = U_K \text{ or } U_KU_IU_K \) by (0.3)). Thus \( K \) is as given.

\( L \) is \( I \)-outer since by (1.7), (1.8) \( V_I(U_K\overline{I}) = (U_{I \circ K} - U_KV_I) \overline{I} \subset U_K\overline{I} \) (by \( K \triangleleft I \)) and \( U_I(U_K\overline{I}) = (U_{I \circ K} + U_{I(K)K} - U_KU_I - (a_{I \circ K} - U_KV_I)\overline{I} \subset U_K\overline{I} \) (by \( K \triangleleft I \)). To see \( L \) is \( I \)-inner it will suffice to show \( U_I\overline{I} \subset K \). \( U_I\overline{I} \) is spanned by all \( U_{x,y} \) and \( U_{x,y} \) for \( s, t \) of the form \( k \text{ or } U_k a \) \( \text{or } k \in K, a \in \overline{I} \). Here \( U_k\overline{I} \). \( U_{U(k)a}\overline{I} = U_{U_k\overline{I}}\overline{I} \subset U_I\overline{I} \) (by (0.3) and \( k \in I \triangleleft J \)). \( U_k\overline{I} \). \( U_{U(k)a}\overline{I} \overline{I} = U_{U_k\overline{I}U_k\overline{I}} \overline{I} \) (by linearized (0.2)) \( \subset U_{x,y} \overline{I} \overline{I} \) for all \( k \triangleleft I \) and \( k \text{ or } U_k a \text{ fall in } K \triangleleft I \).

In order to show \( U_I\overline{I} \overline{K} \subset K \), we need to know how \( U_I + V_{I,j} \) interacts with \( U_I \).

1.10 Lemma. If \( I = I^3 \triangleleft J \) and \( \mathcal{W} \) denotes the unital subalgebra of \( \text{End}(J) \) generated by \( U_I, V_{I,j}, V_I \), then

\[
(1.11) \quad V_{I,j} + V_{I,j} \subset V_{I,j}.
\]
\[(1.12) \quad U_I V_J \subset \mathfrak{M} V_{I,J},\]
\[(1.13) \quad \{U_I + V_{I,J}\} U_J \subset \mathfrak{M} \{U_I + V_{I,J}\} \{V_J + 1\},\]
\[(1.14) \quad 2(U_I + V_{I,J}) V_J^2 \subset \mathfrak{M} \{U_I + V_{I,J}\} \{D_{J,J} + V_J + 1\}.\]

**Proof.** For convenience we may assume \(J = \hat{J}\) is unital and avoid a plethora of hats. Then for (1.11) we have \(V_J = V_{I,J} \subset V_{I,J} \) by (1.1), and \(V_{I,J} = V_{U(I)J} \subset -V_{U(I)J} + V_{I,J} \) (by linearized (1.4)) \(\subset V_{I,J} \) (since \(I < J\)), and, dually, \(V_J \subset V_{I,J} \).

For (1.12) we have for \(x \in I, a \in J\) that \(U_x V_a = U_x U_a \) (by (1.1)) = \(V_x a V_x \) - \(V_{U(x)a} \) (by linearized (1.7)) \(\subset \mathfrak{M} U_{I,J} \subset \mathfrak{M} \{V_J V_J - V_{I,J}\} \) (by (1.13)) \(\subset \mathfrak{M} \{V_{I,J} + V_{I,J} \} \) (by (1.11)). For \(U_I U_J \) in (1.13) we have for \(x, y \in I, a \in J\) that \(V_{x,y} a = U_{a,x} V_{x,a} - U_{U(a)y,x} \) (by (1.6)) \(\subset \mathfrak{M} U_{I,J} - U_{I,J} \subset \mathfrak{M} \{V_J V_J - V_{I,J}\} \) (by (1.11)) \(\subset \mathfrak{M} \{V_{I,J} + V_{I,J} \} \) (by (1.11)).

For \(V_{I,J} U_J \) in (1.14) we have for \(x, y \in I, a \in J\) that \(U_{x,y} V_a = U_{a,x} V_{x,a} - U_{U(a)y,x} \) (by (1.7)) \(\subset \mathfrak{M} U_{I,J} \subset \mathfrak{M} \{V_J V_J - V_{I,J}\} \) (by (1.13)).

1.15 Remark. In some cases a subideal is necessarily an ideal: an idempotent subideal is an ideal. This holds, in particular, if the subideal \(A\) has a covering family of idempotents \(\{e_i\}\) (\(K = \sum U_e K\)), thus if \(K\) has a unit (in this case \(K\) is a direct summand).
Proof. As usual, we may assume \( J \) is unital. By (1.9) we have \( \overline{K} = \sum U_j^nL \) for \( L = K + U_jJ \lhd I \) with \( U_j \overline{I} \subset K \). We will show that in both cases \( U_jU_j \overline{K} \subset L \). This shows \( \overline{K} \) is proper if \( K \) is: if \( \overline{K} = I \), then by idempotence \( I = U_jU_jI = U_jU_j \overline{K} \subset L \) forces \( I = L \), hence \( I = U_jI = U_jI \subset K \) forces \( K = I \). By (0.4) \( \overline{K} \) is invariant under any derivation that \( K \) is.

In the notation of (1.10) we have

\[
(*) \quad U_j \overline{K} = \sum U_jU_j^nL \subset \sum 2 \mathfrak{M}(U_j + V_{I,\gamma})V_j^nL
\]

using (1.13) to convert \( U_j \)'s into \( V_j \)'s, and then using (1.7) to move the resulting \( V_j \)'s to the right. In case (ii) \( K \) (and hence \( L \) too, using (1.7)) is invariant under \( V_j \), so \( (*) \) yields \( U_j \overline{K} \subset \sum \mathfrak{M}(U_j + V_{I,\gamma})L \subset L \) since \( L \lhd I \), and \( U_j \overline{K} \subset L \) in this case. In case (i) \( (*) \) yields only \( U_j \overline{K} \subset \mathfrak{M}(U_j + V_{I,\gamma})(L + V_jL) \): here \( I = 2I \) implies \( U_j + V_{I,\gamma} \subset 2(U_j + V_{I,\gamma}) \), so we can use (1.14) repeatedly to convert higher powers \( V_j^nL \) \( m > 2 \) into lower powers plus \( D_{I,\gamma} \)'s, then using \( [D, V_j] = V_{D,\gamma} \) from (0.4) to move the resulting \( D \)'s to the right, and then absorbing these into \( L \) since \( D(L) = D(K) + D(U_kJ) \subset D(K) + U_kD(J) \) (by (0.4)) \( \subset K + U_kJ = L \) (by the assumed invariance of \( K \) under \( D_{I,\gamma} \)). But then \( U_j \overline{K} \subset \mathfrak{M}(L + V_jL) \subset L + V_jL \) (because \( L \lhd I \) and \( \mathfrak{M}V_j \subset (V_j + 1)\mathfrak{M} \) since \( U_jV_j \subset \mathfrak{M} \) by (1.12), \( V_jV_j \subset \mathfrak{M} \) by (1.11), and \( [V_{I,\gamma}, J] = V_{(I,\gamma)} - V_{I,\gamma} \subset \mathfrak{M} \) by (1.7), (1.2), (1.11), so \( U_jU_j \overline{K} \subset U_j(L + V_jL) \subset \mathfrak{M}L \) (by (1.12)). Thus in both cases we have \( U_jU_j \overline{K} \subset L \), as desired. \( \square \)

In characteristic 2 the maps \( D_{\gamma} = V_{\gamma} \) are derivations too.

2.2 Lemma. If \( 2I = 0 \) then \( V_a \) induces a derivation of \( I \) for each \( a \in J \smallsetminus I \).

Proof. \( V_a \) leaves \( I \lhd J \) invariant, and on \( I \) it satisfies conditions (0.4) since \( D(x^2) = x^2 \circ a = x \circ (x \circ a) - 2U_2a \) (by (1.3)) \( = x \circ (x \circ a) \) (since \( 2U_2a \subset 2I = 0 \)) \( = x \circ D(x) \) and \( D(U_x y) = V_x U_x y = \{ -U_x V_a + U_{a \circ x, \gamma} \} y \) (by (1.7)) \( = U_{V_a} D(y) + U_{D_{(x,\gamma)}} y \) (since \( a \circ y = a \circ y \) for \( y \in I \) because \( 2I = 0 \)). \( \square \)

Now we can prove our main result about subideals of idempotent ideals.

2.3 Theorem. A minimal ideal \( I \lhd J \) in a quadratic Jordan algebra is either trivial (\( I^3 = 0 \)) or \( \mathfrak{P} \)-simple (indeed, has no proper ideals invariant under those inner derivations induced on \( I \) by \( J \)).

Proof. In order to apply 2.1 we must show \( 2I = I \) or \( 0 \); but \( 2I \) is again an ideal of \( J \), so by minimality of \( I \) it can only be \( I \) or \( 0 \). \( \square \)

The heart of a Jordan algebra is the intersection \( \mathfrak{K}(J) = \cap I \) of all nonzero ideals \( I \). If it is nonzero it is the unique minimal ideal, contained in all other ideals, so

2.4 Theorem. The heart \( \mathfrak{K}(J) \) of a quadratic Jordan algebra \( J \) is either trivial or \( \mathfrak{P} \)-simple. \( \square \)

In Zelmanov's work [5] the heart played a major role: the heart of an \( i \)-exceptional prime algebra was actually simple with capacity, from which it follows that the prime \( i \)-exceptional algebra was itself essentially a simple Albert algebra.

The result we want, of course, is that a minimal ideal is trivial or (ordinary) simple. This holds for associative and alternative algebras by work of Slater and
Zhevlakov [4]. It can be deduced for certain Jordan algebras from Block's characterization [1] of differentiably simple algebras.

In [3] it was shown that the middle nucleus and center coincide for -semiprime linear Jordan algebras; since -simple algebras are -semiprime, we get

2.5 Proposition. If is a nontrivial minimal ideal in a linear Jordan algebra, then \( N_m(I) = C(I) \).

References

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