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SIMULTANEOUS ATTAINABILITY OF CENTRAL LYAPUNOV AND BOHL EXPONENTS FOR ODE LINEAR SYSTEMS

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ABSTRACT. Millionščikov's Accessibility Theorem for the central Lyapunov exponent of a linear ODE system is extended to simultaneous attainability of both central Lyapunov and Bohl exponents.

1. Let

$$\dot{x} = A(t)x, \qquad t \geqslant 0, x \in \mathbf{R}^n, \|A(t)\| \leqslant a_0. \tag{1}$$

The *Lyapunov exponent* $\lambda(x)$ and *Bohl exponent* $\beta(x)$ of a solution $x(t)$ are given by

$$\lambda(x) = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln |x(t)|, \quad \text{resp. } \beta(x) = \overline{\lim_{t-s\to\infty}} \frac{1}{t-s} \ln \frac{|x(t)|}{|x(s)|}.$$

(In fact these are *upper* exponents; the lower ones are defined similarly, with $\underline{\lim}$ in place of $\overline{\lim}$.)

In general neither these exponents nor their suprema $\lambda_0 = \sup_x \lambda(x)$, $\beta_0 = \sup_x \beta(x)$ are stable under small perturbations of the system. Instead the so-called *central* Lyapunov exponent[1] $\Lambda \geqslant \lambda_0$ and Bohl exponent $\mathrm{B} \geqslant \beta_0$ can be defined being stable upward (resp. lower exponents being stable downward). To introduce them and to describe exactly this "upward stability" we need a notion of upper functions (for brevity we omit similar notions and results about lower exponents).

2. Let $X(t, s) = X(t)X^{-1}(s)$ where $X(t)$ is a fundamental matrix of (1). As is known,

$$|X(t, s)| \leqslant e^{a_0|t-s|} \tag{2}$$

and

$$|X(t, s)| = \max_x \frac{|x(t)|}{|x(s)|}, \tag{3}$$

where max is taken over all nonzero solutions of (1).

DEFINITION. A bounded function $K(t)$ is an *upper* function for system (1) if there is a constant $D = D_K$ such that

$$|X(t, s)| \leqslant D e^{\int_s^t K(\alpha)\, d\alpha} \qquad (t \geqslant s). \tag{4}$$

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[1]More popular notation is Ω rather than Λ.

For example, by (2), $K(t) = a_0$ is an upper function with $D = 1$. Let

$$\bar{K} = \overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t K(\alpha)\,d\alpha, \qquad \bar{\bar{K}} = \overline{\lim_{t-s\to\infty}} \frac{1}{t-s}\int_s^t K(\alpha)\,d\alpha. \tag{5}$$

DEFINITION. *The central Lyapunov exponent* Λ, resp. *Bohl exponent* B is given by

$$\Lambda = \inf \bar{K}, \quad \text{resp. } \mathrm{B} = \inf \bar{\bar{K}}, \tag{6}$$

where the inf is taken over all upper functions.

It is easily seen that $\lambda_0 \leqslant \Lambda$, $\beta_0 \leqslant \mathrm{B}$ and $\Lambda \leqslant \mathrm{B}$.

3. Consider a perturbed system

$$\dot{y} = \left[A(t) + \tilde{A}(t)\right]y \tag{7}$$

and let its upper functions and exponents be marked by $\tilde{}$.

The upward stability of $K(t)$, Λ, B means that given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that if $|\tilde{A}(t)| \leqslant \delta$, then

$$\tilde{K}(t) \leqslant K(t) + \varepsilon, \qquad \tilde{\Lambda} \leqslant \Lambda + \varepsilon, \qquad \tilde{\mathrm{B}} \leqslant \mathrm{B} + \varepsilon.$$

The next theorem is well known [**1**].

4. THEOREM. $K(t)$, Λ, *and* B *are always upward stable.*

PROOF. It suffices to prove $\tilde{K}(t) \leqslant K(t) + \varepsilon$; then the rest follows by (5), (6). Let $Y(t, s) = Y(t)Y^{-1}(s)$ where $Y(t)$ is a fundamental matrix of (7). By the Variation of Constants Formula,

$$Y(t,s) = X(t,s) + \int_s^t X(t,\tau)\tilde{A}(\tau)Y(\tau,s)\,d\tau.$$

Take norms, use (4) and set

$$|Y(t,s)| = De^{\int_s^t K(\alpha)d\alpha}u(t). \tag{8}$$

Then

$$u(t) \leqslant 1 + \int_s^t D\,|\tilde{A}(\tau)|\,u(\tau)\,d\tau$$

and by Gronwall's inequality, $u(t) \leqslant \exp\int_s^t D\,|A(\tau)|\,d\tau$. Now, if $|\tilde{A}(t)| \leqslant \delta$, then by (8) $\tilde{K}(t) = K(t) + D\delta$ is upper for (7). So $\delta(\varepsilon) = \varepsilon/D$.

In particular Theorem 4 implies that if $\lambda_0 = \lambda$ (or $\beta_0 = \mathrm{B}$), then λ_0 (or β_0) is itself stable up. As is known, for a constant system (1) (i.e. $A(t) = \text{const}$) one has always $\lambda_0 = \beta_0 = \Lambda = \mathrm{B}$, and so all exponents are stable up.

5. In contrast, for nonautonomous systems the central exponents Λ and B need not be attainable by individual solution exponents, i.e. it may happen that $\lambda_0 < \Lambda$ and/or $\lambda_0 < \mathrm{B}$ (as well as $\Lambda < \mathrm{B}$). However the Accessibility Theorem [**2**] states that the central Lyapunov exponent Λ is always attainable by means of arbitrarily small perturbations in the following sense: given $\delta > 0$ there is a perturbation with $|\tilde{A}(t)| < \delta$ such that $\tilde{\lambda}_0 \geqslant \Lambda$ for the perturbed system (7).

It turns out that this theorem can be extended to the attainability of B; moreover, a simultaneous attainability of both Λ and B can be established and at the same time the original proof [**2**] can be considerably shortened.

6. THEOREM. *Let system* (1) *have central Lyapunov exponent* Λ *and Bohl exponent* B. *Given* $\delta_0 > 0$ *there is a perturbation* $\tilde{A}(t)$ *with* $|\tilde{A}(t)| \leqslant \delta_0$ *such that system* (6) *has a solution* $y(t)$ *with both* $\lambda(y) \geqslant \Lambda$ *and* $\beta(y) \geqslant \mathrm{B}$.

To prove this theorem we start with a technical remark and a number of lemmas.

7. REMARK. All the above definitions of exponents or upper functions are given with continuously varying t and s. But nothing will be changed if we replace them by discrete variables $t_n = nT$, $s_m = mT$, where $T > 0$ is fixed and $m, n = 1, 2, \ldots$. This follows by the fact that by (2), $|X(t, s)| \leqslant e^{a_0 T} = \text{const}$ as well as $|x(t)|/|x(s)| \leqslant e^{a_0 T} = \text{const}$ for $|t - s| \leqslant T$, so that any difference between continuous t and discrete $t_n \leqslant t < t_{n+1}$ vanishes by taking $\overline{\lim}$ or else is absorbed by the constant D in (4). In particular, $K(t)$ remains upper if (4) holds just for $t = t_n$, $s = s_m$.

8. LEMMA. *Let* $T > 0$ *be fixed*, $t_n = nT$, $J_n = [t_{n-1}, t_n]$, $n = 0, 1, \ldots$ *and*

$$\ln |X(t, s)| = f(t, s), \quad i.e., \quad |X(t, s)| = e^{f(t,s)}.$$

Define a step function $K(t)$ *by*

$$K(t) \equiv \lambda_n = \frac{1}{T} f(t_n, t_{n-1}) \quad on\ J_n,\ n = 1, 2, \ldots \tag{9}$$

(*the illegal "double definition" at* $t = t_n$ *can be neglected*). *Then* $K(t)$ *is an upper function and hence* $\overline{K} \geqslant \Lambda$, $\overline{\overline{K}} \geqslant \mathrm{B}$.

PROOF. By (2), $K(t)$ is bounded: $|K(t)| \leqslant a_0$. Since $X(t, s) = X(t, r)X(r, s)$, we have $f(t, s) \leqslant f(t, r) + f(r, s)$, and since

$$f(t_k, t_{k-1}) = \lambda_k T = \int_{t_{k-1}}^{t_k} K(\alpha)\, d\alpha,$$

we have for $t = t_n$, $s = t_m$, $n \geqslant m$,

$$f(t, s) \leqslant \Sigma f(t_k, t_{k-1}) = \int_s^t K(\alpha)\, d\alpha, \quad i.e. \quad |X(t, s)| \leqslant e^{\int_s^t K(\alpha)\, d\alpha}.$$

By Remark 7, $K(t)$ is upper.

The next several lemmas constitute so-called Millionščikov's Rotation Method. It can be found in [**2**], that is why we mostly restrict ourselves to some brief outlines of the proofs. Recall that the angle $\gamma = \measuredangle\,(a, b)$ between two vectors $a, b \in \mathbf{R}^n$ is given by $\cos\gamma = a \cdot b/(|a| \cdot |b|)$, $0 \leqslant \gamma \leqslant \pi$.

9. LEMMA. *Let* a *and* c *be vectors in* R^n *with* $|a| = |c|$ *and* $\measuredangle\,(a, c) = \gamma \neq 0, \pi$. *Then there is a unitary operator* $U(t)$: $R^n \to R^n$ *defined on a given interval* J^*: $t^* \leqslant t \leqslant t^* + T$, $T \geqslant 1$, *such that*

(i) $U(t^*) = I$, $U(t^* + T)a = c$,
(ii) $|U(t) - I| = |U^{-1}(t) - I| \leqslant \gamma$,
(iii) $|\dot{U}(t)U^{-1}(t)| \leqslant \gamma$.

SKETCH OF PROOF. Let $V(\omega)$: $R^n \to R^n$ be the rotation by the angle ω from a to c in the 2-plane P_{ac} spanned by a, c, and $V(\omega) = $ identity on the orthogonal complement to P_{ac}. Then $V(\omega)$ is unitary and in a proper orthonormal basis of R^n (the two first elements in P_{ac}) the matrix of $V(\omega)$ is

$$\operatorname{diag}\left\{\begin{pmatrix}\cos\omega & -\sin\omega\\ \sin\omega & \cos\omega\end{pmatrix}, 1, \dots, 1\right\}.$$

Set $U(t) = V[(t - t^*)\gamma/T]$. Then (i) is clear and (ii), (iii) follow by direct computation.

10. LEMMA. *Let $x(t)$ be a solution of* (1) *considered on an interval $J^* = [t^*, t^* + T]$, $T \geqslant 1$. Next, let $x(t^* + T) = a$, and c be a vector with $|c| = |a|$ and $\sphericalangle(a, c) = \gamma \neq 0, \pi$. Then there is a perturbation $\tilde{A}(t)$ with norm*

$$|\tilde{A}(t)| \leqslant \gamma(2a + 1) \tag{10}$$

such that the perturbed system (7) *has a solution $y(t)$ with*

(11)

$$y(t^*) = x(t^*) \quad \textit{and} \quad y(t^* + T) = c \qquad (\textit{so that } |y(t^* + T)| = |x(t^* + T)|).$$

PROOF. Let $y(t) = U(t)x(t)$ where $U(t)$ is as in Lemma 9. Then clearly (11) holds. Next, $\dot{y} = U\dot{x} + \dot{U}x = (UAU^{-1} + \dot{U}U^{-1})y = (A + \tilde{A})y$ where $\tilde{A} = UAU^{-1} - A + \dot{U}U^{-1}$. By Lemma 9, $|\dot{U}U^{-1}| \leqslant \gamma$ and

$$\begin{aligned} |UAU^{-1} - A| &\leqslant |UA(U^{-1} - I)| + |(U - I)A| \\ &\leqslant |UA| \cdot \gamma + \gamma |A| \\ &= 2\gamma |A| \qquad (\text{since } U \text{ is unitary, } |UA| = |A|). \end{aligned}$$

Now (10) follows.

11. LEMMA. *Let a, b, c be three coplanar vectors in R^n such that $|a| = |b| = |c|$ and $0 \leqslant \gamma \leqslant \theta$ where $\gamma = \sphericalangle(a, c)$, $\theta = \sphericalangle(a, b)$. Then $c = \alpha a + \beta b$ where*

$$\beta = \frac{\sin\gamma}{\sin\theta} > 0 \quad \textit{and} \quad \alpha = \frac{\sin(\theta - \gamma)}{\sin\theta} > 0. \tag{12}$$

Proof is by direct computation.

12. PROOF OF THEOREM 6. Choose first γ and T as follows. Let $\delta = \delta_0/2$. Fix γ with

$$0 < \gamma \leqslant \delta/(2a + 1). \tag{13}$$

Then fix $T \geqslant 1$ and so large that $\sin\gamma \geqslant 2e^{-\delta T}$, i.e.

$$\sin\gamma - e^{-\delta T} \geqslant e^{-\delta T}. \tag{14}$$

Define $K(t)$ as in Lemma 8 and classify the solutions $x(t)$ of system (1) on each interval $J_n = [t_{n-1}, t_n]$ as follows.

If

$$\frac{|x(t_n)|}{|x(t_{n-1})|}\begin{cases} = e^{\lambda_n T}, & \text{then } x(t) \text{ is } \textit{maximal} \text{ on } J_n, \\ \geqslant e^{(\lambda_n - \delta)T}, & \text{then } x(t) \text{ is } \textit{rapid} \text{ on } J_n, \\ < e^{(\lambda_n - \delta)T}, & \text{then } x(t) \text{ is } \textit{slow} \text{ on } J_n. \end{cases}$$

Notice that a maximal solution always exists by (3) and (9). Since a constant multiple of $x(t)$ falls into the same category as $x(t)$, we can normalize $x(t)$ as we like without change of its category.

Now we are going to perturb system (1) inductively on each interval $J_1, J_2, \ldots$. Each time the perturbation $\tilde{A}(t)$ will be found by Lemma 10 and hence with $|\tilde{A}(t)| \leqslant \delta$ by virtue of (10) and (13). We will not mention this smallness any longer. Starting with a rapid solution on J_n we will watch its behavior on J_{n+1} and depending on that choose a perturbation on J_n (but not on J_{n+1} yet).

1st *step*. Pick a maximal solution $x(t)$ on J_1. Then it is also rapid

$$\frac{|x(t_1)|}{|x(t_0)|} = e^{\lambda_1 T} > e^{(\lambda_1 - \delta)T} \qquad (t_0 = 0).$$

Look at its natural extension to J_2. If it remains rapid on J_2, i.e.

$$\frac{|x(t_2)|}{|x(t_1)|} \geqslant e^{(\lambda_2 - \delta)T},$$

then put $\tilde{A}(t) \equiv 0$ on J_1, relable $x(t)$ by $y(t)$ on J_1, and the 1st step is completed. As a result we have

$$\text{(15)} \qquad \frac{|y(t_1)|}{|y(t_0)|} \geqslant e^{(\lambda_1 - \delta)T}, \qquad \frac{|x(t_2)|}{|x(t_1)|} \geqslant e^{(\lambda_2 - \delta)T}$$

where $x(t)$ is a natural (unperturbed) extension of $y(t)$.

Suppose $x(t)$ is slow on J_2 and let $x(t_1) = a$. Find a maximal solution $\xi(t)$ on J_2 and normalize it so that the vector $\xi(t_1) = b$ has norm $|b| = |a|$. Since $x(t)$ is slow while $\xi(t)$ is maximal, they cannot be proportional; therefore $\sphericalangle\, (a, b) \neq 0, \pi$. Define a vector c like this: $c = b$ if $\sphericalangle\, (a, b) \leqslant \gamma$, otherwise let $c = \alpha a + \beta b$ be as in Lemma 11.

Now perturb system (1) on J_1 as in Lemma 10. This yields a solution $y(t)$ of (7) with $y(t_0) = x(t_0)$, $|y(t_1)| = |x(t_1)|$ and hence with

$$\frac{|y(t_1)|}{|y(t_0)|} \geqslant e^{(\lambda_1 - \delta)T}.$$

The crucial point is that the natural (unperturbed) extension $z(t)$ of $y(t)$ beyond t_1 is rapid on J_2. Indeed, if $c = b$, then $z(t) = \xi(t)$ which is even maximal on J_2. Otherwise $z(t_1) = c = \alpha a + \beta b = \alpha x(t_1) + \beta \xi(t_1)$ and by linearity

$$z(t) = \alpha x(t) + \beta \xi(t), \qquad t \geqslant t_1.$$

At $t = t_1$ all three norms of z, x, ξ are equal, therefore

$$\text{(16)} \qquad \frac{|z(t_2)|}{|z(t_1)|} = \frac{|\alpha x(t_2) + \beta \xi(t_2)|}{|\xi(t_1)|} \geqslant \frac{|\xi(t_2)|}{|\xi(t_1)|} \left(\beta - \alpha \frac{|x(t_2)|}{|\xi(t_2)|} \right).$$

Since ξ is maximal and x is slow on J_2, we have

$$\frac{|x(t_2)|}{|\xi(t_2)|} = \frac{|x(t_2)|/|x(t_1)|}{|\xi(t_2)|/|\xi(t_1)|} < \frac{e^{(\lambda_2 - \delta)T}}{e^{\lambda_2 T}} = e^{-\delta T}.$$

Then

$$\beta - \alpha\frac{|x(t_2)|}{|\xi(t_2)|} > \frac{\sin\gamma - \sin(\theta-\gamma)e^{-\delta T}}{\sin\theta} \qquad \text{(by (12))}$$

$$\geqslant \sin\gamma - e^{-\delta T}$$

$$\geqslant e^{-\delta T} \qquad \text{(by (14))}.$$

Combining with (15), z is rapid on J_2

$$\frac{|z(t_2)|}{|z(t_1)|} \geqslant e^{(\lambda_2-\delta)T}.$$

Relabeling $z(t)$ by $x(t)$ on J_2, we come again to (15), and the 1st step is entirely completed.

Suppose we have already completed $m-1$ steps of the induction with the following results:

(i) The system is properly perturbed on $J_1 \cup \cdots \cup J_{m-1}$ but unperturbed yet on $J_m = [t_{m-1}, t_m]$ or further.

(ii) There is a solution $y(t)$ of the perturbed system on $J_1 \cup \cdots \cup J_{m-1}$ with natural (unperturbed) continuous extension $x(t)$ on J_m such that

$$\text{(17a)} \qquad \frac{|y(t_k)|}{|y(t_{k-1})|} \geqslant e^{(\lambda_k-\delta)T}, \qquad k = 1,\ldots,m-1,$$

$$\text{(17b)} \qquad \frac{|x(t_m)|}{|x(t_{m-1})|} \geqslant e^{(\lambda_m-\delta)T}.$$

mth *step* is now exactly as the 1st one, just with t_{m-1}, t_m, t_{m+1} in place of t_0, t_1, t_2. Namely, if $x(t)$ remains rapid on J_{m+1} too, then we set $\tilde{A}(t) \equiv 0$ on J_m, relabel $x(t)$ by $y(t)$ on J_m and so get (17a,b) with m replaced by $m+1$. In this case the mth step is completed.

If $x(t)$ is slow on J_{m+1}, then let $x(t_m) = a$, find a maximal solution $\xi(t)$ on J_{m+1} with $\xi(t_m) = b$, $|b| = |a|$, and define c as before: $c = b$ if $\measuredangle\,(a,b) \leqslant \gamma$, otherwise $c = \alpha a + \beta b$ as in Lemma 11. Now perturb the system on J_m as in Lemma 10. This creates a solution $y(t)$ with $y(t_{m-1}) = x(t_{m-1})$, $|y(t_m)| = |x(t_m)|$ and hence, by (17b), with

$$\frac{|y(t_m)|}{|y(t_{m-1})|} \geqslant e^{(\lambda_m-\delta)T}.$$

As before, the unperturbed continuous extension $z(t)$ of $y(t)$ beyond t_m is rapid on J_{m+1}, and relabeling $z(t)$ by $x(t)$ gives again (17a,b) with m replace by $m+1$. The mth step is entirely completed.

By induction, we obtain a system (7) defined for all $t \geqslant 0$ with perturbation $\tilde{A}(t)$ of smallness $|\tilde{A}(t)| \leqslant \delta = \delta_0/2$ and having a solution $y(t)$ which satisfies (17a) for all $k = 1, 2, \ldots$. By the very definition (9) of $K(t)$,

$$\int_{t_{n-1}}^{t_n} K(\alpha)\,d\alpha = \lambda_n T, \qquad \int_{t_{n-1}}^{t_n} [K(\alpha) - \delta]\,d\alpha = (\lambda_n - \delta)T,$$

so that (17a) implies for $s = t_m$, $t = t_n$ $(t \geqslant s)$

$$\frac{|y(t)|}{|y(0)|} \geqslant \exp\int_0^t [K(\alpha) - \delta]\, d\alpha \quad \text{and} \quad \frac{|y(t)|}{|y(s)|} \geqslant \exp\int_s^t [K(\alpha) - \delta]\, d\alpha.$$

It follows by Remark 7 and Lemma 8 that the Lyapunov and Bohl exponents of $y(t)$ satisfy $\lambda(y) \geqslant \overline{K} - \delta \geqslant \Lambda - \delta$ and $\beta(y) \geqslant \overline{\overline{K}} - \delta \geqslant \mathrm{B} - \delta$ respectively. To complete the proof let $y^*(t) = y(t)e^{\delta t}$. Then $\lambda(y^*) \geqslant \Lambda$, $\beta(y^*) \geqslant \mathrm{B}$ and $y^*(t)$ satisfies the system with perturbation $\tilde{A}(t) + \delta I$ of smallness $2\delta = \delta_0$.

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