A PROOF OF A FORMULA
IN FOURIER ANALYSIS ON THE SPHERE

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ABSTRACT. A short and elementary proof of a useful formula in spherical harmonic analysis is provided.

In [3] Sherman proved an integral formula for eigenfunctions of the Laplacian on the sphere \( S^n \). He developed a certain theory of Fourier analysis on the basis of this formula. The purpose of this note is to give an elementary short proof of Sherman's formula.

Let \( a = (0, \ldots, 0, 1) \in S^n \), \( B = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} = 0\} \). \( a \) is the "north pole" and \( B \) is the "equator". For any integer \( k > 0 \) and \( b \in B \), define

\[
e_{b,k}(s) = (a + ib, s)^k, \quad s \in S^n,
\]

and

\[
f_{b,k}(s) = \text{sgn}(s, a)^{n-1}(a + ib, s)^{-k-n+1}, \quad s \in S^n - B,
\]

where \((\cdot, \cdot)\) is the Euclidean inner product, \( i = \sqrt{-1} \). Let \( db \) be the normalized Euclidean measure on \( B \).

**Theorem (Sherman, Lemma 3.9 of [3]).**

\[
\int_B e_{b,k}(s)f_{b,k}(s') \, db = P_k((s, s'))
\]

for all \( s \in S^n \), \( s' \in S^n - B \) and \( k > 0 \), where \( P_k \) is a polynomial of degree \( k \) with \( P_k(1) = 1 \), called the (normalized) Gegenbauer polynomial.

Formula (1) corresponds to formula (1.9) in [3].

**Proof.** Denote the left-hand side of (1) by \( F(s, s') \). Since \( F(-s, -s') = F(s, s') \) we may assume \( \text{sgn}(s', a) = 1 \). Let \( u_\phi \) be the rotation represented by

\[
\begin{bmatrix}
1 \\
\vdots \\
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}.
\]

If \( F(u_\phi s, u_\phi s') = F(s, s') \) for all \( \phi \) such that \( (u_\phi s', a) > 0 \), then the proof of (1) will be reduced to the case \( s' = a \) (in which (1) is the standard integral formula for the
Gegenbauer polynomial, cf. Theorem 7 of [2] or Lemma 4.2 of [3]) because we can always find some rotation $u$ of $B$ and some $u_\phi$ such that $u_\phi u s' = a$, and it is obvious that $F(u s, u s') = F(s, s')$. Hence it is enough to prove $\partial F(u_\phi s, u_\phi s')/\partial \phi = 0$.

Any point $b \in B$ can be written in the form

$$b = (c_1 \cos \theta, c_2 \cos \theta, \ldots, c_{n-1} \cos \theta, \sin \theta, 0)$$

where $c_1^2 + \cdots + c_{n-1}^2 = 1$. Let

$$g(c, \theta, \phi, s) = i \cos \theta \sum_{j=1}^{n-1} c_j s_j + i \sin \theta (s_n \cos \phi - s_{n+1} \sin \phi) + (s_n \sin \phi + s_{n+1} \cos \phi).$$

That means $e_{b,k}(u_\phi s) = [g(c, \theta, \phi, s)]^k$ and $f_{b,k}(u_\phi s') = [g(c, \theta, \phi, s')]^{-k-n+1}$. Then for all integers $m$,

$$\frac{\partial (g^m)}{\partial \theta} = mg^{m-1} \left[ -i \sin \theta \sum_{j=1}^{n-1} c_j s_j + i \cos \theta (s_n \cos \phi - s_{n+1} \sin \phi) \right].$$

$$\frac{\partial (g^m)}{\partial \phi} = mg^{m-1} \left[ i \sin \theta (-s_n \sin \phi - s_{n+1} \cos \phi) + (s_n \cos \phi - s_{n+1} \sin \phi) \right].$$

Hence we obtain a useful relation

$$i \frac{\partial (g^m)}{\partial \phi} - \cos \theta \frac{\partial (g^m)}{\partial \theta} = mg^m \sin \theta.$$ (2)

Therefore

$$\frac{\partial F(u_\phi s, u_\phi s')}{\partial \phi} = \frac{\partial}{\partial \phi} \int_B [g(c, \theta, \phi, s)]^k [g(c, \theta, \phi, s')]^{-k-n+1} db$$

$$= A \int_{c \in S^{n-2}} dc \int_{\pi/2}^{\pi/2} \left[ \frac{\partial (g(c, \theta, \phi, s))^k}{\partial \phi} g(c, \theta, \phi, s')^{-k-n+1} \right] \cos^{n-2} \theta d\theta$$

$$+ g(c, \theta, \phi, s)^k \left[ \cos \theta \frac{\partial (g(c, \theta, \phi, s')^{-k-n+1})}{\partial \phi} \right]^{-k-n+1} + g(c, \theta, \phi, s)^k$$

$$\times \left[ \cos \theta \frac{\partial (g(c, \theta, \phi, s')^{-k-n+1})}{\partial \phi} \right]$$

$$- (k + n - 1) \sin \theta g(c, \theta, \phi, s')^{-k-n+1} \right] \times \cos^{n-2} \theta d\theta$$
by using (2). Here $dc$ is the ordinary Euclidean measure on $S^{n-2}$, $A$ is constant. We can easily see that the integrand is $\partial Q/\partial \theta$, where

$$Q = -i \left[ g(c, \theta, \phi, s)^k g(c, \theta, \phi, s')^{-k-n-1} \cos^{n-1} \theta \right].$$

Therefore the integral is zero.

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