ON A SINGULAR ELLIPTIC EQUATION

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Abstract. In this paper, we study the singular elliptic equation $Lu + K(x)u^p = 0$, where $L$ is a uniformly elliptic operator of divergence form, $p > 1$ and $K(x)$ has a singularity at the origin. We prove that this equation does not possess any positive (local) solution in any punctured neighborhood of the origin if there exist two constants $C_1$, $C_2$ such that $C_1 |x|^a \geq K(x) \geq C_2 |x|^a$ near the origin for some $\sigma \leq -2$ (with no other condition on the gradient of $K$). In fact, an integral condition is derived.

1. Introduction. In recent years, there has been some interest in studying the following singular nonlinear elliptic equation

$$\Delta u + h(x)u^p = 0$$

near the origin in $\mathbb{R}^n$, $n \geq 3$, where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, $p > 1$, and $h$ is nonnegative which may have the origin as an isolated singularity (see, for example, [A], [G, S], [L] and the references therein). In [A], [G, S], $h$ is assumed to satisfy the following conditions (in addition to being positive) near the origin.

$$C_1 |x|^\alpha \geq h(x) \geq C_2 |x|^\sigma$$

for some constants $C_1$, $C_2 > 0$ and $\sigma \in \mathbb{R}$.

Indeed, the asymptotic behavior of positive singular solutions of (1) near the origin are obtained in case $\sigma > -2$ and $p$ in some appropriate ranges (cf. [A], [G, S] and [L]). It is also shown in [G, S] that if $\sigma < -2$, $h$ satisfies (2) and $\nabla \log h$ is bounded near the origin, then equation (1) does not possess any positive solution in $\Omega \setminus \{0\}$, where $\Omega$ is any neighborhood of the origin. The purpose of this present note is to show that equation (1) does not possess any positive solution in $\Omega \setminus \{0\}$ for any $p > 1$ provided $h$ satisfies (2) and $\sigma \leq -2$, i.e. we obtain the same conclusion for all $p > 1$ without any condition on $\nabla h$. Moreover, the exponent $\sigma \leq -2$ is best possible (cf. [A]). In fact, this follows from our main result which applies to more general elliptic operators than Laplacian and the condition on $h$ which insures the above conclusion is an integral condition which actually allows $h$ being "oscillatory" near the origin. We also mention that our proof is simple and elementary in contrast to the complicated proof in [G, S].

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2. The main result. The equation we are studying is

$$(4) \quad Lu + K(x)u^p = 0$$

in $\Omega \setminus \{0\}$, where $\Omega$ is an arbitrary neighborhood of the origin in $\mathbb{R}^n$, $n \geq 3$; $p > 1$, $K > 0$ is smooth in $\Omega \setminus \{0\}$ and

$$L \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} \right)$$

where $a_{ij} \in C^2(\Omega)$, is uniformly elliptic in $\Omega$ (we should point out that the conditions on $(a_{ij})$ may be relaxed without causing any further work later on; see, for example, §2 in [A]).

Let $G(x)$ be the Green's function of $L$ on $\Omega$ with singularity at the origin; we then define

$$\tilde{K}(t) = \min_{x \in \Gamma(t)} K(x)$$

where $\Gamma(t) = \{ x \in \Omega \mid G(x) = t \}$. Now, we can state our main result.

**Theorem.** Under the above hypotheses on $L$ and $K$ equation (4) does not possess any positive solution in any punctured neighborhood of the origin provided

$$(5) \quad \int_{0}^{\infty} t^{-n/(n-2)} \tilde{K}(t) \, dt = \infty.$$

**Proof.** The proof of our theorem uses some averaging method and a result in nonlinear oscillation theory for ordinary differential inequalities in recent papers of Ni [N] and of Aviles [A]. We define

$$\tilde{u}(t) = \int_{\Gamma(t)} \frac{\sum_{i,j=1}^{n} a_{ij} G_{x_i} G_{x_j}}{| \nabla G(x) |} u(x) \, dS$$

as in [A]. Then differentiating $\tilde{u}$ twice with respect to $t$, applying Green's theorem several times, we arrive at the following inequality,

$$(6) \quad \tilde{u}_{tt} + C \frac{\tilde{K}(t)}{t^{2(n-1)/(n-2)}} \tilde{u}^p(t) \leq 0$$

for $t \geq T$, where $C$ is some (fixed) positive constant (see Lemma 1 in §2 of [A] for details). Now, applying Theorem 3.43 in [N] to (6), we conclude our proof. Q.E.D.

**Remarks.** (7) Since $G(x)$ behaves like $C/|x|^{n-2}$ near the origin (by the assumption on $L$), we see that (5) is fulfilled if $K(x)$ behaves like $|x|^\sigma$, $\sigma \leq -2$, near $x = 0$. In fact, since only the total "weight" of $K$ matters, $K$ does not have to have $\infty$ as its limit at $x = 0$, it could have zero as its limit inf at the origin.

**Note added in proof.** The conclusion of Theorem 3.43 in [N] is not quite correct as it stands, what is really proved there is that a solution cannot be ultimately positive, which is just sufficient for the application in [N] and is just what we need in the proof of the main result here. In fact this result in nonlinear oscillation was proved earlier by H. Teufel, Jr., *A note on second-order differential equations and functional

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