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THE SINGULAR INTEGRAL CHARACTERIZATION OF H^p ON SIMPLE MARTINGALES

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ABSTRACT. The singular integral characterization of H^1 on simple martingales was given by S. Janson. We show that his result cannot be extended to H^p if p (> 0) is very small.

Let $\Omega = (0, 1]$. Let F be the σ-field of all Borel sets in Ω. Let dx be the Lebesgue measure. Then (Ω, F, dx) is a probability space. Let $d \geqslant 3$ be an integer. For each integer $n \geqslant 0$, let F_n be the sub-σ-field of F generated by $((k-1)d^{-n}, kd^{-n}]$, $k = 1, \ldots, d^n$. Set

$$I(k_1, \ldots, k_n) = \big((k_1 - 1)d^{-1} + \cdots + (k_{n-1} - 1)d^{1-n} + (k_n - 1)d^{-n},\\ (k_1 - 1)d^{-1} + \cdots + (k_n - 1)d^{1-n} + k_n d^{-n}\big]$$

for each $k_1, \ldots, k_n \in \{1, \ldots, d\}$.

A martingale is a sequence of complex-valued integrable functions $\{f_n\}_{n=0}^{\infty}$ such that $E[f_{n+1} \mid F_n] = f_n$, where $E[\cdot \mid F_n]$ denotes the conditional expectation with respect to the sub-σ-field F_n. We write f for $\{f_n\}_{n=0}^{\infty}$. If f is generated from a function $\tilde{f}(x) \in L^1(\Omega)$ by

$$f_n = E[\tilde{f} \mid F_n], \tag{1}$$

we identify $\tilde{f}$ and f.

For a martingale f we define

$$f^*(x) = \sup_{n \geqslant 0} |f_n(x)|.$$

For $p \in (0, \infty)$, f is said to belong to H^p if $\| f^* \|_p < +\infty$, where

$$\|f^*\|_p = \left\{ \int_\Omega f^*(x)^p \, dx \right\}^{1/p}.$$

It is well known that if $p > 1$, then H^p and $L^p(\Omega)$ can be identified. That is, $f^* \in L^p(\Omega)$ if and only if there exists a function $\tilde{f}(x) \in L^p(\Omega)$ such that (1) and

$$c_p \|f^*\|_p \leqslant \|\tilde{f}\|_p \leqslant \|f^*\|_p, \quad \text{where } c_p > 0,$$

hold. It is also known that if $f^* \in L^1(\Omega)$, then f is generated from an L^1-function but that the converse is not true.

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For $n \geqslant 1$, set $\Delta f_n = f_n - f_{n-1}$. Since Δf_n is F_n-measurable, the notation $\Delta f_n(I(k_1,\ldots,k_n))$ makes sense. Let

$$V = \left\{ \underset{\sim}{x} = (x_k)_{k=1}^d \in \mathbf{C}^d : \sum_{k=1}^d x_k = 0 \right\},$$

where $(x_k)_{k=1}^d$ denotes a d-dimensional column vector and $\mathbf{C}$ is the set of all complex numbers. Note that $(\Delta f_n(I(k_1,\ldots,k_{n-1},k)))_{k=1}^d \in V$. Let A be a linear operator from V to V. Set

$$(\Delta g_n(I(k_1,\ldots,k_{n-1},k)))_{k=1}^d = A(\Delta f_n(I(k_1,\ldots,k_{n-1},k)))_{k=1}^d,$$

$$g_n = \sum_{k=1}^n \Delta g_k, \qquad g_0 = 0,$$

(2) $$Tf = \{g_n\}_{n=0}^\infty \quad \text{and} \quad (Tf)_n = g_n.$$

Assume that $A_1,\ldots,A_m$ are one or more linear transformations on V, and let $T_1,\ldots,T_m$ be the corresponding operators defined by (2).

The above definitions are due to Janson [**9**] and Chao and Taibleson [**4**]. They introduced the above as the analogues of Hardy spaces and singular integral operators on $\mathbf{R}^n$. These have been known to lead to some rewarding feed-backs with the Euclidean case. (See [**8, 9** and **12**].)

For $p \leqslant 1$ and close to 1, the above definitions work very well. Janson [**9**] showed

THEOREM A. *There exists $p_0(A_1,\ldots,A_m) < 1$ such that*

(3) $$\liminf_{n\to\infty} \left\{ \|f_n\|_p + \sum_{j=1}^m \left\|(T_j f)_n\right\|_p \right\} \geqslant C_{p,A_1,\ldots,A_m} \|f^*\|_p$$

for any $p > p_0$ and any martingale f if and only if

(4) $A_1,\ldots,A_m$ *do not have a common real eigenvector, where* $C_{p,A_1,\ldots,A_m} > 0$.

(See also Chao and Taibleson [**4**].)

The hard implication is the "if" part. Janson proved this from the observation that if (4) holds, then $(|f_n|^2 + \sum_{j=1}^m |(T_j f)_n|^2)^{p_0/2}$ becomes a submartingale. (For another argument for the case $p = 1$ that does not appeal to the submartingale property, see [**11**].)

In this note, we show

THEOREM 1. *There exists $p_1(d) > 0$ such that*

$$\inf \left\{ \limsup_{n\to\infty} \left(\|f_n\|_p + \sum_{j=1}^m \left\|(T_j f)_n\right\|_p \right) \Big/ \|f^*\|_p : f \in H^p, f \not\equiv 0, f_0 = 0 \right\} = 0$$

for any $p \in (0, p_1(d)]$, any $m \geqslant 1$ and any $A_1,\ldots,A_m$, where $p_1(d)$ depends only on d.

As a consequence of Theorem 1, $p_0(A_1,\ldots,A_m)$ in Theorem A cannot be less than $p_1(d)$, no matter how we choose m and $A_1,\ldots,A_m$. Thus, a "singular integral" characterization of H^p on these simple martingales is impossible for $p \leqslant p_1(d)$ if we define "singular integrals" by (2). This tells us that H^p theory on these martingales and that on $\mathbf{R}^n$ are not quite parallel if p is very small.

We get Theorem 1 as a corollary of the following Theorem 2. Let $\mathbf{H}$ be a Hilbert space. Let

$$\mathbf{V} = \left\{ (\mathbf{x}_k)_{k=1}^d : \mathbf{x}_k \in \mathbf{H},\ \sum_{k=1}^d \mathbf{x}_k = \mathbf{0} \right\}.$$

Let $\mathbf{A}$ be a linear operator from V to $\mathbf{V}$. Set

$$(5)\qquad (\Delta \mathbf{g}_n(I(k_1,\ldots,k_{n-1},k)))_{k=1}^d = \mathbf{A}(\Delta f_n(I(k_1,\ldots,k_{n-1},k)))_{k=1}^d.$$

Set

$$\mathbf{g}_0 = \mathbf{0},\quad \mathbf{g}_n = \sum_{k=1}^n \Delta \mathbf{g}_k \quad \text{and} \quad \mathbf{g} = \{\mathbf{g}_n\}_{n=0}^\infty.$$

Then $\mathbf{g}$ is an $\mathbf{H}$-valued martingale. Set $\mathbf{T}f = \mathbf{g}$ and $(\mathbf{T}f)_n = \mathbf{g}_n$.

THEOREM 2. *There exists $p_1(d) > 0$ such that*

$$\inf\left\{ \limsup_{n\to\infty} \|(\mathbf{T}f)_n\|_p / \|f^*\|_p : f \in H^p, f \not\equiv 0, f_0 = 0 \right\} = 0.$$

for any $p \in (0, p_1(d)]$ and any $\mathbf{A}$, where $p_1(d)$ depends only on d.

For $\mathfrak{x} = (x_k)_{k=1}^d \in V$, set $\mathfrak{x}(k) = x_k$. If we substitute $\mathbf{H} = \mathbf{C}^{m+1}$ and

$$\mathbf{A}\mathfrak{x} = \begin{pmatrix} (\mathfrak{x}(1), (A_1\mathfrak{x})(1), \ldots, (A_m\mathfrak{x})(1)) \\ (\mathfrak{x}(2), (A_1\mathfrak{x})(2), \ldots, (A_m\mathfrak{x})(2)) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (\mathfrak{x}(d), (A_1\mathfrak{x})(d), \ldots, (A_m\mathfrak{x})(d)) \end{pmatrix},$$

then $\mathbf{A}$ is a linear operator from V to $\mathbf{V}$ and

$$(\mathbf{T}f)_n = (f_n, (T_1 f)_n, \ldots, (T_m f)_n).$$

Thus, Theorem 1 follows from Theorem 2.

We now prove Theorem 2.

We define integer-valued functions

$$\{h_i(k_1,\ldots,k_n)\}_{n=1,2,3,\ldots;\, i,k_1,\ldots,k_n \in \{1,\ldots,d\}}$$

inductively by

$$(6)\qquad h_i(k) = \delta(i,k),$$

$$(7)\qquad h_i(k_1,\ldots,k_{n-1},k) = h_i(k_1,\ldots,k_{n-1}) - h_{k_{n-1}}(k_1,\ldots,k_{n-1})\delta(i,k),$$

where $\delta(i,k)$ is the Kronecker delta.

Set

$$\mathfrak{x}^0 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in V, \qquad (\Delta f_1(I(k)))_{k=1}^d = \mathfrak{x}^0,$$

and

$$(\Delta f_n(I(k_1,\ldots,k_{n-1},k)))_{k=1}^d = -h_{k_{n-1}}(k_1,\ldots,k_{n-1})\mathfrak{x}^0, \quad \text{for } n \geqslant 2.$$

From this Δf_n we define $\Delta \mathbf{g}_n$ by (5). Then

$$(\Delta \mathbf{g}_1(I(k)))_{k=1}^{d} = \mathbf{A}\mathfrak{x}^0 \tag{8}$$

and

$$(\Delta \mathbf{g}_n(I(k_1,\ldots,k_{n-1},k)))_{k=1}^{d} = -h_{k_{n-1}}(k_1,\ldots,k_{n-1})\mathbf{A}\mathfrak{x}^0, \quad \text{for } n \geqslant 2. \tag{9}$$

Set $(\mathbf{x}_k^0)_{k=1}^d = \mathbf{A}\mathfrak{x}^0$. Then by (6)–(9),

$$\mathbf{g}_n(I(k_1,\ldots,k_n)) = \sum_{i=1}^{d} h_i(k_1,\ldots,k_n)\mathbf{x}_i^0, \quad \text{for } n \geqslant 1.$$

(For $n = 1$ this is clear from (6) and (8). Assume that this holds for $n - 1$. Then by (9) and (7),

$$\begin{aligned}\mathbf{g}_n(I(k_1,\ldots,k_n)) &= \mathbf{g}_{n-1}(I(k_1,\ldots,k_{n-1})) + \Delta\mathbf{g}_n(I(k_1,\ldots,k_n))\\ &= \sum_{i=1}^{d} h_i(k_1,\ldots,k_{n-1})\mathbf{x}_i^0 - h_{k_{n-1}}(k_1,\ldots,k_{n-1})\mathbf{x}_{k_n}^0\\ &= \sum_{i=1}^{d} h_i(k_1,\ldots,k_n)\mathbf{x}_i^0.).\end{aligned}$$

Note that it follows from (7) that

$$\begin{aligned}&h_{k_{n-1}}(k_1,\ldots,k_{n-2},k_{n-1},k_{n-1}) = 0,\\ &h_i(k_1,\ldots,k_{n-2},k_{n-1},k_{n-1}) = h_i(k_1,\ldots,k_{n-2},k_{n-1}) \quad \text{if } i \neq k_{n-1},\\ &h_i(k_1,\ldots,k_{n-2},k_{n-1},k_{n-1},j) = h_i(k_1,\ldots,k_{n-2},k_{n-1},k_{n-1})\\ &\qquad\qquad \text{for any } i,\ j \in \{1,\ldots,d\} \text{ and } n \geqslant 2.\end{aligned}$$

Thus, if $k_1 = k_2,\ k_3 = k_4,\ldots,\ k_{2d-1} = k_{2d}$, and if $\{k_1, k_3,\ldots,k_{2d-1}\} = \{1,2,\ldots,d\}$, then

$$h_i(k_1,k_2,\ldots,k_{2d-1},k_{2d}) = 0 \quad \text{for any } i \in \{1,\ldots,d\}. \tag{10}$$

Set

$$\mathfrak{C}_n = \{I(k_1,\ldots,k_n) \colon h_i(k_1,\ldots,k_n) = 0 \text{ for any } i \in \{1,\ldots,d\}\}.$$

By (10),

$$\sum_{I\in\mathfrak{C}_{2d}} |I| \geqslant d^{-2d}\cdot d!.$$

Repeating this estimate, we get

$$\begin{aligned}\sum_{I\in\mathfrak{C}_{2nd}} |I| &\geqslant d^{-2d}\cdot d!\Big\{1 - \sum_{I\in\mathfrak{C}_{2(n-1)d}} |I|\Big\} + \sum_{I\in\mathfrak{C}_{2(n-1)d}} |I|\\ &= d^{-2d}\cdot d! + (1 - d^{-2d}\cdot d!)\sum_{I\in\mathfrak{C}_{2(n-1)d}} |I|\\ &\geqslant d^{-2d}\cdot d!\{1 + (1 - d^{-2d}\cdot d!) + \cdots + (1 - d^{-2d}\cdot d!)^{n-1}\}\\ &= 1 - (1 - d^{-2d}\cdot d!)^n.\end{aligned} \tag{11}$$

On the other hand, from (6)–(7), $|h_i(k_1,\ldots,k_n)| \leqslant 2^{n-1}$. Thus,

$$\|\mathbf{g}_n\|_\infty \leqslant C_{\mathrm{A}} 2^{n-1}. \tag{12}$$

Hence by (11)–(12),

$$\int_\Omega |\mathbf{g}_{2nd}|^p\, dx = \int_{\{\mathbf{g}_{2nd}\neq 0\}} |\mathbf{g}_{2nd}|^p\, dx \leqslant \left(C_{\mathrm{A}} 2^{2nd}\right)^p (1 - d^{-2d}\cdot d!)^n \to 0 \quad \text{as } n \to \infty$$

if $p > 0$ is small enough that

$$2^{2dp}(1 - d^{-2d}\cdot d!) < 1. \tag{13}$$

But since $f_n^*(x) \geqslant 1$ on $I(1)$,

$$\int f_n^*(x)^p\, dx \geqslant d^{-1}.$$

Thus,

$$\|\mathbf{g}_n\|_p / \|f_n^*\|_p \to 0 \quad \text{as } n \to \infty$$

if (13) holds. This concludes the proof of Theorem 2.

REMARK. In Chao [2], a conjecture about the best possible p_0 in Theorem A for the case $d = 3$ is given.

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