A FAMILY OF POLYNOMIALS WITH CONCYCLIC ZEROS

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Abstract. Expand \( E(z) = (e^z - 1)^m \) by the binomial theorem, and replace every \( \exp(kz) \) by its approximation \( (1 + kn^{-1}z)^n \). The resulting polynomial has all of its zeros on a circle of radius \( r \) centered at \(-r\), where \( r = n/m \).

1. Introduction. For positive integers \( n \) let \( P_n \) be the linear mapping from the exponential polynomials over \( C \) to the polynomials over \( C \) that replaces \( \exp(az) \) by

\[
\left(1 + \frac{az}{n}\right)^n,
\]

but is otherwise the identity. For example,

\[
P_n\left((e^z - 1)^2\right) = \left(1 + \frac{2z}{n}\right)^n - 2\left(1 + \frac{z}{n}\right)^n + 1.
\]

Thus \( P_n \) applied to any exponential polynomial \( E(z) \) would be the identity.

Our purpose is to establish Conjecture 2 of [14]. This is now

**Theorem 1.** Let \( n \) and \( m \) be positive integers with \( n \geq m \). The zeros of

\[
P_n((e^z - 1)^m) = \left(1 + \frac{mz}{n}\right)^n - \ldots
\]

all lie on \( C(r) \), the circle of radius \( r \) centered at \(-r\), where \( r = n/m \).

Set

\[
w = 1 + \frac{mz}{n}.
\]

It suffices to show that \( w \) or \( w^{-1} \) lies on the unit circle \( U \). But this follows from the binomial expansion and Theorem 3.

The proof of Theorem 3 relies on exploiting a certain linear fractional transformation and upon

**Theorem 2 (A. Cohn, 1922).** If \( P(w) \) is a real polynomial of degree \( n \) and

\[
w^nP(w^{-1}) = \pm P(w),
\]

then the zeros of \( P \) all lie on \( U \) if and only if the zeros of its derivative all lie on or inside \( U \).

For Theorem 2 (and a bit more) see [3 or 6, p. 206].
2. The proof. We now establish

\textbf{Theorem 3.} Let \( n \geq m \). Then the polynomial

\begin{equation}
R_m(w) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} [kw + (m - k)]^n
\end{equation}

has all its roots on \( U \).

\textbf{Proof.} It is clear for \( m = 1 \). Assume the result for \( m - 1 \). Since

\begin{equation}
R'_m(w) = n \sum_{k=0}^{m} (-1)^k \binom{m}{k} k[kw + (m - k)]^{n-1},
\end{equation}

it suffices by A. Cohn’s theorem to show that the zeros of

\begin{equation}
S_m(w) = \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{k}{m} [kw + (m - k)]^{n-1}
\end{equation}

have modulus at most 1. Let

\begin{equation}
z = \frac{mw}{w + m - 1} \quad \text{so} \quad w = \frac{(m - 1)z}{m - z}.
\end{equation}

Then

\begin{equation}
kw + (m - k) = m \cdot \frac{(k - 1)z + m - k}{m - z},
\end{equation}

so

\begin{equation}
\left( \frac{m - z}{m} \right)^{n-1} S_m(w) = \sum_{k=1}^{m} (-1)^k \binom{m}{k-1} [(k - 1)z + m - k]^{n-1}
\end{equation}

\begin{equation}
= - \sum_{k=0}^{m-1} (-1)^k \binom{m - 1}{k} [kz + (m - 1 - k)]^{n-1}.
\end{equation}

By the induction hypothesis, the vanishing of \( S_m(w) \) implies that \( z \) is on \( U \). Hence

\begin{equation}
|w| \leq \frac{(m - 1)z}{m - |z|} = 1
\end{equation}

and the result follows.

We remark that by making further linear fractional transformations one can show that the roots of

\begin{equation}
\sum_{k=0}^{m} (-1)^k \binom{m}{k} [(k - 1)z - (k - 2)]^n = 0
\end{equation}

lie on a circle about \( (m - 3)/(m - 2) \) of radius \( 1/(m - 2) \), and that for any positive integer \( h \) the roots of

\begin{equation}
\sum_{k=0}^{m} (-1)^k \binom{m}{k} [(k - h)z + (m - k - h)]^n
\end{equation}

lie on \( U \). For the study of a more difficult family of polynomials of which (2.1) is a special case when \( m = 2 \), see [7].
Remarks. The paper [14] shows how this fits into the general program of “reductionism”. The author's previous proof of the theorem (for \( m = 4 \) only) involved the much heavier machinery of [7]. For further material on zeros of exponential polynomials see, e.g., [1, pp. 120–122], [2, 4, 5, 8, 9, 12, 15], and the many references therein.

For another way in which linear fractional transformations enter into proofs of similar results, see [10] and the idea of Dyson in [11].

In general, if \( E(z) \) is an exponential polynomial with all zeros real, then the zeros of \( P_n E(z) \) lie in a narrow annulus; see [14].

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References


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