PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 4, August 1983

A FAMILY OF POLYNOMIALS WITH CONCYCLIC ZEROS

KENNETH B. STOLARSKY

ABSTRACT. Expand $E(z) = (e^z - 1)^m$ by the binomial theorem, and replace every $\exp(kz)$ by its approximation $(1 + kn^{-1}z)^n$. The resulting polynomial has all of its zeros on a circle of radius r centered at $-r$, where $r = n/m$.

1. Introduction. For positive integers n let P_n be the linear mapping from the exponential polynomials over $\mathbf{C}$ to the polynomials over $\mathbf{C}$ that replaces $\exp(az)$ by

$$\left(1 + \frac{az}{n}\right)^n, \tag{1.1}$$

but is otherwise the identity. For example,

$$P_n\{(e^z - 1)^2\} = \left(1 + \frac{2z}{n}\right)^n - 2\left(1 + \frac{z}{n}\right)^n + 1.$$

Thus P_∞ applied to any exponential polynomial $E(z)$ would be the identity.

Our purpose is to establish Conjecture 2 of [**14**]. This is now

THEOREM 1. *Let n and m be positive integers with $n \geqslant m$. The zeros of*

$$P_n\{(e^z - 1)^m\} = \left(1 + \frac{mz}{n}\right)^n - \cdots \tag{1.2}$$

all lie on $C(r)$, the circle of radius r centered at $-r$, where $r = n/m$.

Set

$$w = 1 + \frac{mz}{n}. \tag{1.3}$$

It suffices to show that w or w^{-1} lies on the *unit circle U*. But this follows from the binomial expansion and Theorem 3.

The proof of Theorem 3 relies on exploiting a certain linear fractional transformation and upon

THEOREM 2 (A. COHN, 1922). *If $P(w)$ is a real polynomial of degree n and*

$$w^n P(w^{-1}) = \pm P(w), \tag{1.4}$$

then the zeros of P all lie on U if and only if the zeros of its derivative all lie on or inside U.

For Theorem 2 (and a bit more) see [**3** or **6**, p. 206].

Received by the editors October 25, 1982.

1980 *Mathematics Subject Classification*. Primary 30C15; Secondary 33A10.

Key words and phrases. Concyclic zeros, exponential, exponential polynomial, linear fractional transformation, "reductionism", zeros of polynomials.

0002-9939/82/0000-1318/$02.00

2. The proof. We now establish

THEOREM 3. *Let $n \geqslant m$. Then the polynomial*

$$R_m(w) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} [kw + (m-k)]^n \tag{2.1}$$

has all its roots on U.

PROOF. It is clear for $m = 1$. Assume the result for $m - 1$. Since

$$R'_m(w) = n \sum_{k=0}^{m} (-1)^k \binom{m}{k} k[kw + (m-k)]^{n-1}, \tag{2.2}$$

it suffices by A. Cohn's theorem to show that the zeros of

$$S_m(w) = \sum_{k=1}^{m} (-1)^k \binom{m}{k} \frac{k}{m} [kw + (m-k)]^{n-1} \tag{2.3}$$

have modulus at most 1. Let

$$z = \frac{mw}{w+m-1} \quad \text{so } w = \frac{(m-1)z}{m-z}. \tag{2.4}$$

Then

$$kw + (m-k) = m \cdot \frac{(k-1)z + m - k}{m-z}, \tag{2.5}$$

so

$$\begin{aligned} \left(\frac{m-z}{m}\right)^{n-1} S_m(w) &= \sum_{k=1}^{m} (-1)^k \binom{m-1}{k-1} [(k-1)z + m - k]^{n-1} \\ &= -\sum_{k=0}^{m-1} (-1)^k \binom{m-1}{k} [kz + (m-1-k)]^{n-1}. \end{aligned} \tag{2.6}$$

By the induction hypothesis, the vanishing of $S_m(w)$ implies that z is on U. Hence

$$|w| \leqslant \frac{(m-1)|z|}{m-|z|} = 1 \tag{2.7}$$

and the result follows.

We remark that by making further linear fractional transformations one can show that the roots of

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} [(k-1)z - (k-2)]^n = 0 \tag{2.8}$$

lie on a circle about $(m-3)/(m-2)$ of radius $1/(m-2)$, and that for any positive integer h the roots of

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} [(k-h)z + (m-k-h)]^n \tag{2.9}$$

lie on U. For the study of a more difficult family of polynomials of which (2.1) is a special case when $m = 2$, see [7].

REMARKS. The paper [**14**] shows how this fits into the general program of "reductionism". The author's previous proof of the theorem (for $m = 4$ only) involved the much heavier machinery of [**7**]. For further material on zeros of exponential polynomials see, e.g., [**1**, pp. 120–122], [**2, 4, 5, 8, 9, 12, 15**], and the many references therein.

For another way in which linear fractional transformations enter into proofs of similar results, see [**10**] and the idea of Dyson in [**11**].

In general, if $E(z)$ is an exponential polynomial with all zeros real, then the zeros of $P_n E(z)$ lie in a narrow annulus; see [**14**].

ACKNOWLEDGEMENT. The author thanks Kevin McCurley for stimulating discussions and much assistance with computer generation of numerical examples.

REFERENCES

1. A. Baker, *Transcendental number theory*, Cambridge Univ. Press, Cambridge, 1975.
2. R. E. Bellman and K. L. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.
3. A. Cohn, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Z. **14** (1922), 110–148.
4. D. G. Dickson, *Zeros of exponential sums*, Proc. Amer. Math. Soc. **16** (1965), 84–89.
5. R. E. Langer, *On the zeros of exponential sums and integrals*, Bull. Amer. Math. Soc. **37** (1931), 213–239.
6. M. Marden, *Geometry of polynomials*, Math. Surveys, No. 3, Amer. Math. Soc., Providence, R.I., 1966.
7. J. D. Nulton and K. B. Stolarsky, *The zeros of a certain family of trinomials*, in preparation.
8. A. J. van der Poorten, *On the number of zeros of functions*, Enseign. Math. (2) **23** (1977), 19–38.
9. A. J. van der Poorten and R. Tijdeman, *On common zeros of exponential polynomials*, Enseign. Math. (2) **21** (1975), 57–67.
10. D. Ruelle, *Some remarks on the location of zeros of the partition function for lattice systems*, Comm. Math. Phys. **31** (1973), 265–277.
11. ______, *Extension of the Lee-Yang circle theorem*, Phys. Rev. Lett. **26** (1971), 303–304.
12. E. Schwengeler, *Geometrisches über die Verteilung der Nullstellen spezieller ganzer funktionen*, Thesis, Zürich, 1925.
13. K. B. Stolarsky, *Zero-free regions for exponential sums*, Proc. Amer. Math. Soc. **83** (1981), 486–488.
14. ______, *Zeros of exponential polynomials and "reductionism"*, Colloq. Math. Soc. János Bolyai, vol. 34, Topics in Classical Number Theory, Elsevier, New York; North-Holland, Amsterdam (to appear).
15. M. Voorhoeve, *On the oscillation of exponential polynomials*, Math. Z. **151** (1976), 277–294.

DEPARTMENT OF MATHEMATICS, 1409 WEST GREEN, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801