PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 4, August 1983

# INFINITE-DIMENSIONAL JACOBI MATRICES ASSOCIATED WITH JULIA SETS

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ABSTRACT. Let $B$ be the Julia set associated with the polynomial $Tz = z^N + k_1 z^{N-1} + \cdots + k_N$, and let $\mu$ be the balanced $T$-invariant measure on $B$. Assuming $B$ is totally real, we give relations among the entries in the infinite-dimensional Jacobi matrix $J$ whose spectral measure is $\mu$. The specific example $Tz = z^3 - \lambda z$ is given, and some of the asymptotic properties of the entries in $J$ are presented.

**1. Introduction.** Let $C$ be the complex plane and $T$: $C \to C$ a polynomial, $T(z) = z^N + k_1 z^{N-1} + \cdots + k_N$ where $N \geq 2$ and each $k_i \in C$. Define $T^0(z) = z$ and $T^n(z) = T \circ T^{n-1}(z)$ for $n \in \{1, 2, 3, \ldots\}$. A fundamental role in the study of the sequence of iterates $\{T^n(z)\}$ is played by the Julia set $B$. $B$ is the set of points $z \in C$ where $\{T^n(z)\}$ is not normal in the sense of Montel, and a general exposition can be found in Julia [**8**], Fatou [**6, 7**] and Brolin [**5**]. It has positive logarithmic capacity, and on it can be placed an equilibrium charge distribution $\mu$. This provides a measure on $B$ which is invariant under $T$: $B \to B$ and is such that the system $(B, \mu, T)$ is strongly mixing.

In an earlier paper [**1**] we investigated general properties of $\mu$ and its associated orthogonal monic polynomials. Here we restrict attention to the case where $B$ is a compact subset of the real line, and the orthogonal polynomials satisfy a three-term recurrence formula. In [**2**] we proved, for $N = 2$, relationships connecting the coefficients, which permit all the polynomials to be calculated in a recursive fashion. Here we generalized the relationships so that the orthogonal polynomials of all degrees can be obtained for any $T$ for which $B$ is a compact subset of the real line (Theorem 1). The results are illustrated for $T(z) = z^3 - \lambda z$ with $\lambda \geq 3$. When $\lambda = 3$ the polynomials are those of Chebychev, shifted to the interval $[-2, 2]$, and when $\lambda > 3$ they become a generalization whose support is a Cantor set. In this case we establish that both the coefficients (Theorem 2) and the associated Jacobi matrix $J$ (Theorem 3) display an asymptotic self-reproducing property.

**2. Preliminaries.**

DEFINITION 1. $\mu$ is a balanced $T$-invariant Borel measure on $B$ if $\mu$ is a probability measure supported on $B$, such that for any complete assignment of branches of $T^{-1}$, namely $T_j^{-1}$ for $j \in \{1, 2, 3, \ldots, N\}$, $\mu(T_j^{-1}(S)) = \mu(S)/N$ for each Borel set $S$.

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Received by the editors March 23, 1982.
1980 *Mathematics Subject Classification*. Primary 30C10; Secondary 47B25.
[1]Supported by NSF grant MCS-8104862.
[2]Supported by NSF grant MCS-8002731.


0002-9947/82/0000-1166/$02.00

There is only one balanced $T$-invariant measure on $B$, and the equilibrium measure of Brolin is balanced [3]. If $\mu$ is balanced and $f \in L^1(B, \mu)$, then [1]

(1) $$\langle z^j f(T(z))\rangle = S_j \langle f(z)\rangle / N \quad \text{for } j \in \{1, 2, \ldots, N-1\},$$

where $\langle f(z)\rangle = \int_B f(z)\, d\mu(z)$. Here

(2) $$S_j = -jk_j - \sum_{l=1}^{j-1} k_l S_l$$

with $k_l$ the coefficient of $Z^{N-l}$ in $T$ for $l \in \{1, 2, \ldots, N\}$.

In [1] we showed that the sequence of monic polynomials $\{P_n(z)\}_{n=0}^{\infty}$, orthogonal with respect to $\mu$ according to $\langle \overline{P_l(z)} P_m(z)\rangle = 0$ for $l \neq m$, obey the following relations:

(a) $P_1(z) = z + k_1/N$,
(b) $P_{lN}(z) = P_l(T(z))$ for $l \in \{0, 1, 2, \ldots\}$,
(c) $P_{N^l}(z) = T^l(z) + k_1/N$ for $l \in \{0, 1, 2, \ldots\}$.

**3. Results.** When $B$ is a subset of the real line the orthonormal polynomials with respect to $\mu$ obey (b) and the following relation.

(3) $$a(n+1)p_{n+1}(x) + b(n)p_n(x) + a(n)p_{n-1}(x) = xp_n(x), \quad n \in \{0, 1, 2, \ldots\},$$
$$p_{-1}(x) = 0, \qquad p_0(x) = 1,$$

where

$$a(n) = \langle xp_n p_{n-1}\rangle \quad \text{for } n \in \{1, 2, 3, \ldots\},$$

and

$$b(n) = \langle xp_n^2\rangle \quad \text{for } n \in \{0, 1, 2, \ldots\}.$$

The recurrence formula (3) can be recast as the formal operator equation

(4) $$J\psi = x\psi$$

where

(5) $$J = \begin{bmatrix} b(0) & a(1) & 0 & \cdots \\ a(1) & b(1) & a(2) & \cdots \\ 0 & a(2) & b(2) & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

and $\psi^T = (p_0, p_1, p_2, \ldots)$. $J$ can be treated as a selfadjoint operator acting on $l_2$. In [2] we showed that the coefficients in $J$ obey certain recurrence formulas when $T$ is quadratic; see also [4]. We generalize that result here.

Proceeding formally we have

(6) $$J^l\psi = x^l\psi \quad \text{for } l \in \{0, 1, 2, \ldots\},$$

which leads to

(7) $$\langle p_{nN} J^l \psi \hat{e}_{nN+1}\rangle = \langle x^l p_{nN}^2\rangle$$

for $n \in \{0, 1, 2, \dots\}$, where $\hat{e}_k$ is the $l_2$ vector with one in the $k$th place and zeros elsewhere. Observe also that the invariance of $\mu$ together with (b) implies

$$\begin{aligned}(8) \qquad a(n) &= \langle x p_n p_{n-1} \rangle = \langle T(x) p_{nN} p_{nN-N} \rangle \\ &= a(nN) a(nN-1) \cdots a(nN-N+1), \qquad n \in \{1, 2, 3, \dots\}.\end{aligned}$$

THEOREM 1. *Let* $a(n) = b(n-1) = 0$ *for* $n \leqslant 0$. *Then all of the coefficients in* $J$ *can be calculated recursively using* (8) *and* (7) *with* $l \in \{1, 2, \dots, 2N-1\}$.

The proof will require two lemmas.

LEMMA 1. *Let* $\{p_n\}_0^\infty$ *be the orthonormal polynomials associated with the balanced* $T$*-invariant* $\mu$. *Then*

$$(9) \qquad \langle x^l p_{nN}^2 \rangle = D(l) \quad \text{for } l \in \{1, 2, \dots, 2N-1\},$$

*where*

$$D(l) = \begin{cases} N^{-1} S_l & \text{when } l \in \{1, 2, \dots, N-1\}, \\ N^{-1} S_{l-N} b(n) - \sum_{j=1}^{N} k_j D(l-j) & \text{when } l \in \{N, \dots, 2N-1\}, \end{cases}$$

*where* $S_0 = N$ *and* $S_l$ *is otherwise as defined in* (2).

PROOF OF LEMMA 1. For $l \in \{1, 2, \dots, N-1\}$ the result follows from (1) with $f = p_{nN}^2$. For $l = N + m$,

$$(10) \qquad x^{N+m} = x^m T(x) - \sum_{j=1}^{N} k_j x^{m+N-j}.$$

The lemma now follows on multiplying through by $p_{nN}^2$, integrating, and using the fact that

$$(11) \qquad \langle x^m T(x) p_{nN}^2 \rangle = N^{-1} S_m \langle x p_n^2 \rangle = N^{-1} S_m b(n)$$

for $m \in \{0, 1, 2, \dots, N-1\}$.

One can now see that the dependence on $n$ on the right-hand side enters only through $b(n)$.

LEMMA 2. *Let* $C^l(nN+1, nN+1)$ *denote the* $(nN+1, nN+1)$ *entry in* $J^l$. *When* $l = 2k$, *the coefficient in* $C^{2k}(nN+1, nN+1)$ *with the highest index is* $a(nN+k)$ *and all other coefficients have lower indices. When* $l = 2k+1$, *the coefficients in* $C^{2k+1}(nN+1, nN+1)$ *with the highest index are* $a(nN+k)$ *and* $b(nN+k)$; *all other coefficients have lower indices.*

PROOF OF LEMMA 2. We begin by computing $C^l(nN+1, nN+1)$ with the aid of (7). Thus

(12)
$$\begin{aligned} C^l(nN+1, nN+1) &= a(nN) C^{l-1}(nN, nN+1) + b(nN) C^{l-1}(nN+1, nN+1) \\ &\quad + a(nN+1) C^{l-1}(nN+2, nN+1), \qquad l \in \{1, 2, \dots, 2N-1\}, \end{aligned}$$

with

$$(13) \qquad C^1(i, j) = a(i-1)\delta_{i-1,j} + b(i-1)\delta_{i,j} + a(i)\delta_{i+1,j},$$

and

$$C^m(i,j) = a(i-1)C^{m-1}(i-1,j)b(i-1) + C^{m-1}(i,j) + a(i)C^{m-1}(i+1,j). \tag{14}$$

It follows immediately from (14) that $C^m(i,j) = 0$ if $|i-j| > m$. From (13) and (14) we find

$$C^1(nN+1, nN+1) = b(nN), \tag{15}$$

and

$$C^2(nN+1, nN+1) = a(nN)^2 + b(nN)^2 + a(nN+1)^2. \tag{16}$$

Let us now assume that the lemma holds up to $2k-1$. Then

$$C^{2k}(nN+1, nN+1) = a(nN+1)C^{2k-1}(nN+2, nN+1) + b(nN)C^{2k-1}(nN+1, nN+1) + a(nN)C^{2k-1}(nN+1, nN+1). \tag{17}$$

One can easily show by induction that if $a(l)$ or $b(n)$ appear in $C^m(i,j)$ then $l \leqslant (m+i+j)/2$ and $n \leqslant (m+i+j-1)/2$. Consequently one need only consider the first term on the right-hand side of (17). Therefore

$$C^{2k}(nN+1, nN+1) = \left[\prod_{l=1}^{k} a(nN+l)\right] C^k(nN+k+1,\ nN+1)$$
$$+ \{\text{terms containing only coefficients with indices lower than } nN+k\}.$$

But from (14) we have

$$C^k(nN+k+1, nN+1) = \prod_{l=1}^{k} a(nN+l), \tag{18}$$

whence

(19)

$$C^{2k}(nN+1, nN+1) = \left[\prod_{l=1}^{k} a(nN+l)\right]^2$$
$$+ \{\text{terms involving only coefficients with indices lower than } nN+k\}.$$

Likewise,

$$C^{2k+1}(nN+1, nN+1) = \left[\prod_{l=1}^{k} a(nN+l)\right] C^{k+1}(nN+k+1, nN+1) \tag{20}$$
$$+ \{\text{terms involving only } a(l) \text{ and } b(l-1) \text{ with } l < nN+k\},$$

and (14) now yields

$$C^{2k+1}(nN+1, nN+1) = \left[\prod_{l=1}^{k} a(nN+l)\right]^2 b(nN+k) \tag{21}$$
$$+ \{\text{terms involving only } a(l) \text{ and } b(l-1) \text{ with } l < nN+k\}.$$

This completes the proof of Lemma 2.

PROOF OF THEOREM 1. If one is given $a(i)$ and $b(i)$ for $i < Nn$, then Lemmas 1 and 2, together with (8), provide $2N$ relations from which one can explicitly calculate $a(nN + l)$ and $b(nN + l)$ for $l \in \{0, 1, 2, \ldots, N - 1\}$. This completes the proof.

COROLLARY 1. *If $B$ is an interval on the real line then $B = [a, b]$ with $a = -k_1/N - 2$ and $b = -k_1/N + 2$. Moreover, $d\mu = dx/\pi\{(b - x)(x - a)\}^{1/2}$, and $T(x) + k_1/N$ is the monic Chebychev polynomial of degree $N$ on $B$.*

PROOF. If $B$ is an interval then the electrical equilibrium distribution $\mu$ is just the measure associated with the Chebychev polynomials of the first kind. Since all the off-diagonal entries in $J$ except for $a(1)$ are the same, (6) implies these must equal unity. Likewise, all diagonal entries in $J$ must be equal to $-k_1/N$, and the proof is completed.

**4. An example.** We examine the case $T(z) = z^3 - \lambda z$ with $\lambda \geqslant 3$, for which Theorem 1 yields

$$b(n) = 0, \tag{22}$$

$$a(3n + 1)^2 = 2\lambda/3 - a(3n)^2, \tag{23}$$

$$a(3n + 2)^2 = \lambda/3 \tag{24}$$

and

$$a(3n)a(3n - 1)a(3n - 2) = a(n). \tag{25}$$

From these relations and Corollary 1 it is easy to see that $B = [-2, 2]$ when $\lambda = 3$. For $\lambda > 3$ it follows from [**5**] that $B$ is a totally disconnected perfect subset of the real line, with Lebesgue measure zero. As such, it is a generalized Cantor set.

LEMMA 3. *For $\lambda > 3$ and $n \in \{1, 2, 3, \ldots\}$, $0 < a(3n) < 1$ and $a(3n) < a(n)$.*

PROOF. From (23) and (25) it follows that $a(1)^2 = 2\lambda/3$ and $a(3)^2 = 3/\lambda$. Furthermore, from (23)–(25) we have

$$a(3n)^2 = \frac{3}{\lambda}\frac{a(n)^2}{2\lambda/3 - a(3n - 3)^2}, \tag{26}$$

and the lemma follows by induction and equations (23) and (24).

THEOREM 2. *For $\lambda > 3$ and $m, s \in \{0, 1, 2, \ldots\}$,*

$$\operatorname*{Lim}_{n\to\infty} a(m3^n + s)^2 = a(s)^2.$$

PROOF. First consider the case $s = 0$. Then from (26)

$$\begin{aligned} a(m3^n)^2 &= (3/\lambda)a(m3^{n-1})^2/\left(2\lambda/3 - a(m3^n - 3)^2\right) \\ &< (3/\lambda)a(m3^{n-1})^2/(2\lambda/3 - 1) < (3/\lambda)^n(2\lambda/3 - 1)^{-n}a(m)^2. \end{aligned}$$

Because $3/\lambda < 1$, and $2\lambda/3 - 1 > 1$, for $\lambda > 3$ we now have $\operatorname{Lim}_{n\to\infty} a(m3^n)^2 = 0$. The proof is now completed by induction on $m$ for $s = 3m + k$, $k \in \{0, 1, 2, \ldots\}$, using (23)–(25).

Results similar to Lemma 3 and Theorem 2 are valid for $T(z) = (z - \lambda)^2$ with $\lambda \geqslant 2$ and follow from **[2]**; see, for example, **[4]**.

Now consider the sequence of infinite-dimensional Jacobi matrices $\{J^{(m3^n)}\}$ defined for $m, n \in \{0, 1, 2, \ldots\}$ by

$$J^{(m3^n)} = \begin{pmatrix} 0 & a(m3^n+1) & 0 & \cdot \\ a(m3^n+1) & 0 & a(m3^n+2) & \cdot \\ 0 & a(m3^n+2) & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Here the coefficients $a(i)$ are those determined by (23)–(25). Since the support $B$ of the spectral measure of $J$ is compact, it also is for each $J^{(m3^n)}$, and, hence, each matrix corresponds to a selfadjoint operator in $l_2$.

THEOREM 3. *For each $m \in \{0, 1, 2, \ldots\}$ and $\lambda \geqslant 3$ the sequence of operators $\{J^{(m3^n)}\}_{n=0}^{\infty}$ converges strongly to $J$.*

This theorem, and indeed Theorem 2 also, are immediate when $\lambda = 3$ because then

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot \\ 1 & 0 & 1 & 0 & \cdot \\ 0 & 1 & 0 & 1 & \cdot \\ 0 & 0 & 1 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

PROOF OF THEOREM 3. Since the spectrum of $J$ is compact, the entries of $J^{(m3^n)}$ are uniformly bounded. The result now follows since the weak convergence implied by Theorem 2 implies the strong operator convergence

$$\lim_{n \to \infty} \|(J - J^{(m3^n)})x\| = 0, \quad \text{for all } x \in l_2, \tag{27}$$

for banded matrices. This completes the proof.

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