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ON FUNCTIONS THAT APPROXIMATE RELATIONS

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ABSTRACT. Let X be a metric space and let Y be a separable metric space. Suppose R is a relation in $X \times Y$. The following are equivalent: (a) for each $\varepsilon > 0$ there exists f: $X \to Y$ such that the Hausdorff distance from f to R is at most ε; (b) the domain of R is a dense subset of X, and for each isolated point x of the domain the vertical section of R at x is a singleton; (c) for each $\varepsilon > 0$ there exists f: $X \to Y$ of Baire class one such that the Hausdorff distance from f to R is at most ε.

Let $\langle X, d_X \rangle$ and $\langle Y, d_Y \rangle$ be metric spaces. By a *relation* R in $X \times Y$ we mean a nonempty subset of the product. Let us make $X \times Y$ a metric space by defining the distance ρ between points (x_1, y_1) and (x_2, y_2) in the product by

$$\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

A function f: $X \to Y$ will be said [3] to *ε-approximate* R if each point in f has ρ-distance at most ε from some point in R, and each point of R has ρ-distance at most ε from some point in f. Alternatively, f can be called [2] an *ε-approximate selection* for R, although this terminology has been used differently by Michael [7] and Deutsch and Kenderov [4]. More formally, if f is an ε-approximate selection for R, then f has *Hausdorff distance* at most ε from R. We now pause to describe this notion.

Let W be a metric space. For each point w in W let $S_\varepsilon[w]$ denote the open ball of radius ε with center w in W. If $C \subset W$ denote $\bigcup_{w \in C} S_\varepsilon[w]$ by $S_\varepsilon[C]$. If K is another set in W and there exists $\varepsilon > 0$ for which both $S_\varepsilon[C] \supset K$ and $S_\varepsilon[K] \supset C$, then the Hausdorff distance δ between C and K is given by

$$\delta[C, K] = \inf\{\varepsilon\colon S_\varepsilon[C] \supset K \text{ and } S_\varepsilon[K] \supset C\}.$$

If no such ε exists, we write $\delta[C, K] = \infty$. Further information on this notion of distance can be found in Aubin [1], Kuratowski [6], or Nadler [9]. Now if δ denotes Hausdorff distance in $X \times Y$ as induced by ρ and R is a nonempty subset of $X \times Y$ and f: $X \to Y$, then the symbol $\delta[f, R]$ makes sense, and it is clear that (i) if f ε-approximates R, then $\delta[f, R] \leqslant \varepsilon$; (ii) if $\delta[f, R] \leqslant \varepsilon$ then f θ-approximates R for each $\theta > \varepsilon$.

The main purpose of this note is to characterize for arbitrary X and separable Y those relations in $X \times Y$ that admit for each $\varepsilon > 0$ a Borel ε-approximate selection. We shall in fact show that the existence for each $\varepsilon > 0$ of an ε-approximate selection

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(Borel measurable or not) for the relation implies the existence for each $\varepsilon > 0$ of a Baire class one ε-approximate selection.

DEFINITION. Let X and Y be metric spaces. A function $f\colon X \to Y$ is said to be of *Baire class* $\alpha < \Omega$ if for each open subset G of Y the set $f^{-1}(G)$ is of additive class α in X.

In particular, $f\colon X \to Y$ is of Baire class one if the inverse image of each open subset of Y is an F_σ subset of X. For a thorough discussion of such functions, the reader should consult Kuratowski [**6**], where the functions of Baire class α are called *B-measurable of class* α. We need two results from this source, which we state as lemmas. The first is not deep; the second is a serious theorem of Montgomery [**8**].

LEMMA A. *Let X and Y be metric spaces. Suppose $\{A_i\colon i \in Z^+\}$ is a collection of sets each of additive class α with union X. Suppose $f\colon X \to Y$ and for each $i \in Z^+$ the restriction of f to A_i is of Baire class α. Then f is of Baire class α.*

LEMMA B. *Let X be a metric space and let $F \subset X$. Suppose for each $x \in X$ there exists an open neighborhood V_x of x such that $F \cap V_x$ is of additive class α. Then F itself is of additive class α.*

Since open sets in a metric space are F_σ sets, the phrase "$F \cap V_x$ is of additive class α" used in Lemma B is unambigous: subsets of V_x that are of additive class α with respect to the relative topology on V_x are precisely those that are of additive class α with respect to the topology on X. In the sequel we shall use the following notation for the *domain* and *vertical section* at x of a relation R in $X \times Y$:

$$\mathrm{Dom}(R) = \{x\colon \text{for some } y, (x, y) \in R\}, \quad R(x) = \{y\colon (x, y) \in R\}.$$

THEOREM 1. *Let X be a metric space and Y a separable metric space. Suppose R is a relation in $X \times Y$. The following are equivalent.*

(a) *For each $\varepsilon > 0$ there exists $f\colon X \to Y$ such that $\delta[f, R] \leqslant \varepsilon$.*

(b) *The domain of R is a dense subset of X, and for each isolated point x of the domain the section $R(x)$ is a singleton.*

(c) *For each $\varepsilon > 0$ there exists $f\colon X \to Y$ of Baire class one such that $\delta[f, R] \leqslant \varepsilon$.*

PROOF. (a) $\to$ (b). Suppose that $\overline{\mathrm{Dom}(R)} \neq X$. Then there exists $x \in X$ and $\varepsilon > 0$ such that $S_\varepsilon[x] \cap \mathrm{Dom}(R) = \varnothing$. It follows that if $f\colon X \to Y$ is arbitrary, then $f \not\subset S_\varepsilon[R]$, whence $\delta[f, R] \geqslant \varepsilon$. Suppose now that $\overline{\mathrm{Dom}(R)} = X$, but for some isolated point x of $\mathrm{Dom}(R)$ the section $R(x)$ contains two distinct points y_1 and y_2 of Y. Since x must be an isolated point of X, there exists $\varepsilon > 0$ such that both $d_Y(y_1, y_2) > \varepsilon$ and $S_\varepsilon[x] = \{x\}$. Hence if $f\colon X \to Y$ satisfies $\delta[f, R] < \varepsilon/2$ we must simultaneously have $d_Y(f(x), y_1) < \varepsilon/2$ and $d_Y(f(x), y_2) < \varepsilon/2$, an impossibility.

(b) $\to$ (c). If X has no limit points, then $\mathrm{Dom}(R) = X$ and each vertical section of R is a singleton. Thus, R is a continuous function, and there is nothing to prove. Otherwise, let $\theta = \varepsilon/2$ and let L denote the set of limit points of X. Consider the family Ω of subsets S of L with the following property: for each $\{x, z\} \subset S$, $d_X(x, z) \geqslant \theta$. If Ω is partially ordered by inclusion, then by Zorn's lemma Ω has a maximal member, say, $\{x_i\colon i \in I\}$, and it easily follows that $L \subset \bigcup_{i \in I} S_\theta[x_i]$. Let

$C = \{y_n\colon n \in Z^+\}$ be a countable dense subset of Y, and for each $i \in I$ choose $y_{n(i)} \in C$ whose distance from $\bigcup\{R(x)\colon x \in S_\theta[x_i] \cap \mathrm{Dom}(R)\}$ is less than θ. Since $W = \bigcup_{i \in I} S_\theta[x_i]$ as a subspace of X is paracompact and regular, there is an open refinement $\{V_\lambda\colon \lambda \in \Lambda\}$ of the cover $\{S_\theta[x_i]\colon i \in I\}$ of W such that $\{\overline{V}_\lambda\colon \lambda \in \Lambda\}$ is a locally finite (closed) refinement of $\{S_\theta[x_i]\colon i \in I\}$. Let $E = \bigcup_{i \in I} S_{\theta/3}[x_i]$. We first define a Baire class one function on this open subspace of X. Let $i \in I$ be arbitrary. Since x_i is a limit point of X, there is a sequence $\{x_{ni}\}$ of distinct points in $S_{\theta/3}[x_i]$ convergent to x_i. We define $h_i\colon S_{\theta/3}[x_i] \to Y$ as follows: let h_i map $\{x_{ni}\colon n \in Z^+\}$ onto a dense subset of $\bigcup\{R(x)\colon x \in S_\theta[x_i] \cap \mathrm{Dom}(R)\}$, and let h_i assign to each remaining point of $S_{\theta/3}[x_i]$ the point $y_{n(i)}$. Functions with countable domains are automatically of Baire class one; so $h_i \mid \{x_{ni}\colon n \in Z^+\}$ and $h_i \mid S_{\theta/3}[x_i] - \{x_{ni}\colon n \in Z^+\}$ are both of Baire class one. Since the set $\{x_{ni}\colon n \in Z^+\}$ and its complement in $S_{\theta/3}[x_i]$ are both F_σ sets, Lemma A implies that h_i itself is of Baire class one. Now set $h = \bigcup_{i \in I} h_i$. By the construction of $\{x_i\colon i \in I\}$, it is clear that h is a well-defined function from E to Y. Since the inverse image of each open set under h is locally an F_σ set, Lemma B ensures that h is of Baire class one.

We next define a Baire class one function on $W - E$. For each $\lambda \in \Lambda$ choose $i(\lambda) \in I$ such that $\overline{V}_\lambda \subset S_\theta[x_{i(\lambda)}]$. For each $x \in W - E$ let $g(x) = y_{n(i(\lambda))}$ where $n(i(\lambda))$ is the smallest integer such that $x \in \overline{V}_\lambda$. We claim that for each $m \in Z^+$ the set $g^{-1}(\{y_1, y_2, \ldots, y_m\})$ is a relatively closed subset of $W - E$. To see this let $\{w_k\}$ be a sequence in $g^{-1}(\{y_1, y_2, \ldots, y_m\})$ convergent to some point w of $W - E$. Since $\{\overline{V}_\lambda\colon \lambda \in \Lambda\}$ is locally finite there exist indices $\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \subset \Lambda$ and an integer K such that for each $k > K$, $\{\lambda\colon \lambda \in \Lambda \text{ and } w_k \in \overline{V}_\lambda\} \subset \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$. Now for each $k > K$ there exists $\lambda(k) \in \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ such that $w_k \in \overline{V}_{\lambda(k)}$ and $n(i(\lambda(k))) \leqslant m$. Since $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ is finite, there exists $\lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ for which $w \in \overline{V}_\lambda$ and $n(i(\lambda)) \leqslant m$. This establishes the claim. Since the intersection of a closed set with an open set is an F_σ set, for each $m \geqslant 2$,

$$g^{-1}(\{y_m\}) = g^{-1}(\{y_1, \ldots, y_m\}) - g^{-1}(\{y_1, \ldots, y_{m-1}\})$$

is a relatively F_σ subset of $W - E$. Hence for each open set G of Y the set $g^{-1}(G) = g^{-1}(C \cap G)$ is a relatively F_σ subset of $W - E$, and it follows that $g\colon W - E \to Y$ is of Baire class one.

On $X - W$ the relation R reduces to a continuous function. Since the sets E, $W - E$, and $X - W$ are each F_σ subsets of X, by Lemma A the function $f\colon X \to Y$ defined by

$$f(x) = \begin{cases} h(x) & \text{if } x \in E, \\ g(x) & \text{if } x \in W - E, \\ R(x) & \text{if } x \in X - W \end{cases}$$

is of Baire class one. It remains to show that $\delta[f, R] \leqslant \varepsilon = 2\theta$. We first show that each point in f is within ε of some point in R. If $x \in X - W$ then $(x, f(x)) \in R$. If $x \in W - E$ then there exists $i \in I$ such that $x \in S_\theta[x_i]$ and $f(x) = y_{n(i)}$. However, by the definition of $y_{n(i)}$ there exists a point x_i^* in $S_\theta[x_i]$ and a point $y \in R(x_i^*)$ for which $d_Y(y, y_{n(i)}) < \theta$. It follows that $\rho[(x, f(x)), (x_i^*, y)] < \max\{2\theta, \theta\} = \varepsilon$. Finally, if $x \in E$ then there exists $i \in I$ such that $d_X(x, x_i) < \theta/3$; moreover, $f(x)$ is either $y_{n(i)}$ or a point in $\cup\{R(z)\colon z \in S_\theta[x_i] \cap \mathrm{Dom}(R)\}$. In either case $(x, f(x))$ has ρ-distance less than ε from some point (x_i^*, y), where $x_i^* \in S_\theta[x_i]$ and $y \in R(x_i^*)$. We now must show that each point of R is within ε of some point of f. If $x \in X - W$ then $R(x)$ is a singleton and $R(x) = f(x)$. Next let $x \in W \cap \mathrm{Dom}(R)$ and choose $y \in R(x)$. There exists $i \in I$ such that $x \in S_\theta[x_i]$. Recall, however, that $\{f(x_{ni})\colon n \in Z^+\}$ is dense in $\cup\{R(z)\colon z \in S_\theta[x_i] \cap \mathrm{Dom}(R)\}$, so there exists $n \in Z^+$ for which $d_Y(f(x_{ni}), y) < \varepsilon$. Again, it is clear that $\rho[(x, y), (x_{ni}, f(x_{ni}))] < \varepsilon$, and this portion of the proof is complete.

(c) $\to$ (a). Obvious.

Theorem 1 fails without the separability assumption on Y.

EXAMPLE 1. Let X be the rationals, viewed as a subspace of the line with the usual topology, and let Y be an uncountable set with the discrete metric. Let $R = X \times Y$. Now each $f\colon X \to Y$ has a countable range, and it follows from the definition of the metric ρ on $X \times Y$ that $\delta[f, R] = 1$.

Following Michael we could call $f\colon X \to Y$ an ε-approximate selection for a relation R with domain X if, for each x in X, $f(x) \in S_\varepsilon[R(x)]$. The existence of Baire class one approximate selections in this context would seem to rest on some continuity requirement on the map $x \to R(x)$. For example, the property of *almost lower semicontinuity*, due to Deutsch and Kenderov [**4**], is sufficient [**2**]: for each x in X there exists a neighborhood V_x of x such that $\cap\{S_\varepsilon[R(w)]\colon w \in V_x\}$ is nonempty.

EXAMPLE 2. Let $X = Y = [0, 1]$ and let B be a non-Borel set in the interval. Let $R \subset X \times Y$ be the characteristic function of the set B. Then if $f\colon X \to Y$ is a $\frac{1}{3}$-approximate selection for R (in the sense of Michael), then $f^{-1}((\frac{1}{2}, 1]) = B$, a non-Borel set. Thus, R admits no Borel $\frac{1}{3}$-approximate selection.

Continuous approximate selections, either in our sense or that of Michael, can be obtained for certain well-behaved relations with convex vertical sections. A recent example: if X is paracompact and Y is a normed linear space, then those relations with domain X that admit for each $\varepsilon > 0$ a continuous ε-approximate selection in the sense of Michael are precisely those that are almost lower semicontinuous [**4**]. Invariably, such approximations are constructed by piecing together continuous functions defined locally via a partition of unity [**5**, p. 170] to yield a globally defined continuous function that is close to the relation. We close by showing that locally defined Baire class α functions are subject to such an amalgamation, provided X is metric and Y is a second countable topological vector space.

THEOREM 2. *Let X be a metric space and let Y be a second countable topological vector space. Let $\{U_\lambda\colon \lambda \in \Lambda\}$ be a locally finite open cover of X and let $\{p_\lambda(\cdot)\colon \lambda \in \Lambda\}$ be a partition of unity subordinated to the cover. Suppose for each $\lambda \in \Lambda$ the function $f_\lambda\colon U_\lambda \to Y$ is of Baire class α. Then $f\colon X \to Y$ defined by $f(x) = \sum_{\lambda \in \Lambda} p_\lambda(x) f_\lambda(x)$ is of Baire class α.*

PROOF. Fix x in X and let V_x be an open neighborhood of x that meets only finitely many members of the open cover, say $\{U_{\lambda_1}, \ldots, U_{\lambda_n}\}$. By Lemma B we need only show that $f | V_x$ is of Baire class α. Now for each z in V_x we have $f(z) = \sum_{i=1}^{n} p_{\lambda_i}(z) f_{\lambda_i}(z)$. Since the restriction of each function of Baire class α is of Baire class α on its restricted domain, to show that $f | V_x$ is of Baire class α it suffices to show that if $h_1: X \to Y$ and $h_2: X \to Y$ are of Baire class α and p is a real valued continuous function on X, then both ph_1 and $h_1 + h_2$ are of Baire class α. We prove the former statement, leaving the latter to the reader. Let $\{G_i: i \in Z^+\}$ and $\{U_i: i \in Z^+\}$ be bases for the topologies on Y and the line, respectively. Consider ϕ: $X \to Y \times R$ defined by $\phi(x) = (h_1(x), p(x))$. Since $\phi^{-1}(G_i \times U_j) = h_1^{-1}(G_i) \cap p^{-1}(U_j)$ and the sets of additive class α contain the open sets and are closed under finite intersections and countable unions, the second countability of $Y \times R$ implies $\phi^{-1}(G)$ is of additive class α for each open set G in the product. Since ψ: $Y \times R \to Y$ defined by $\psi(y, \theta) = \theta y$ is continuous, $ph_1 = \psi \circ \phi$ is of Baire class α.

It is important to note that Theorem 2 cannot be used to piece together locally Borel functions to obtain a globally Borel function. Using the well-known example of Szpilrajn-Marczewski [**6**] of a non-Borel set in a metric space that is nevertheless locally Borel, a counterexample can be easily constructed. The details are left to the reader.

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