A NOTE ON THE STRONG MAXIMAL FUNCTION

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Abstract. Given a nonnegative measurable function $f$ on $R^2$ which is integrable over sets of finite measure, we construct a new function $g$ with the same distribution function as $f$ such that the strong maximal function of $g$ has the same local integrability properties as its Hardy-Littlewood maximal function.

The Hardy-Littlewood maximal operator in $R^n$ is defined by

$$Mf(x) = \sup \left\{ \frac{1}{|Q|} \int_{Q} |f| : x \in Q \right\},$$

where $Q$ is a cube in $R^n$ with edges parallel to the coordinate axes. For $f$ with bounded support, well-known arguments show that $Mf$ is locally integrable provided $\int |f| \log^+ |f|$ is integrable; E. M. Stein [4] proved that this condition on $f$ is also necessary.

The strong maximal function $M_n f$ is defined in $R^n$ similarly; the cubes are replaced by rectangles of arbitrary shape but oriented with edges parallel to the coordinate axes. Jessen, Marcinkiewicz, and Zygmund [3] noted $M_n f$ could be dominated by the composition of one-dimensional maximal operators; accordingly, $M_n f$ is integrable over sets of finite measure provided $\int |f| \left( \log^+ |f| \right)^n$ is integrable, and no weaker local integrable condition on $f$ is sufficient.

It has been conjectured by Fava, Gatto, and Gutiérrez [2] that this condition is also necessary; we became interested in this problem during the preparation of [1]. We show that the conjecture is false by constructing a rich class of functions on $R^2$ for which averages over rectangles are dominated by averages over squares. Consequently, local integrability properties of $M_2 f$ imply nothing stronger than the identical properties for $Mf$.

Theorem. Let $f$ be a nonnegative measurable function on $R^2$ which is integrable over sets of finite measure. Then there is a function $g$ on $R^2$ such that

$$| \{ x : f(x) > \lambda \} | = | \{ x : g(x) > \lambda \} | , \quad 0 < \lambda < \infty,$$

and

$$M_2 g(x) \leq \frac{5}{4} Mg\left( \frac{x}{3} \right) + \sup_{|E| \leq 1} \int_{E} g , \quad a.e. \ x \in R^2.$$
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PROOF. First we introduce some notation. We define

\[ Q = \{ x \in \mathbb{R}^2 : |x_1| \leq 3 \text{ and } |x_2| \leq 3 \}, \]

\[ L_t = \{ x \in \mathbb{R}^2 : x_2 - x_1 = t \}, \]

\[ D_t = \{ x \in \mathbb{R}^2 : |x_2 - x_1| \leq t \}, \]

\[ A_t = \{ x \in D_t : |x_1 + x_2| \leq 2 \}. \]

For \( x \notin L_0 \), we let \( Q_x \) be the square with one corner at \( x \) and one diagonal along \( L_0 \).

We define a function \( t(\lambda) \), \( 0 < \lambda < \infty \), by

\[ t(\lambda) = \frac{1}{\lambda} \left| \left\{ x \in \mathbb{R}^2 : f(x) > \lambda \right\} \right| \]

so that \( |A_{t(\lambda)}| = \left| \left\{ x : f(x) > \lambda \right\} \right| \). We now define our function \( g \) by

\[ g(x) = \int_{t_0}^{\infty} \chi_\lambda(x) \, d\lambda, \]

where \( \chi_\lambda \) is the characteristic function of \( A_{t(\lambda)} \). It is then clear that the first condition on \( g \) is satisfied; we direct our attention to estimating \( M_2 g \).

For the remainder of our argument, \( x \) will be a fixed point not in \( L_0 \) and \( R \) will be a generic rectangle containing \( x \) and having its sides parallel to the coordinate axes. We have

\[ \int_R g = \int_0^{\infty} |A_{t(\lambda)} \cap R| \, d\lambda. \]

Choosing \( \lambda_0 = \inf \{ \lambda : |A_{t(\lambda)}| \leq 1 \} \), we use \( |A_{t(\lambda)} \cap R| \leq |R| \) for \( \lambda < \lambda_0 \) and show that for \( |A_t| \leq 1 \),

\[ (*) \quad 4 |A_t \cap R| / |R| \leq |A_t| + 5 |Q_x \cap 3A_t| / |Q_x|. \]

Assuming the validity of \((*)\),

\[ \frac{1}{|R|} \int_R g \leq \lambda_0 + \frac{1}{4} \int_{\lambda_0}^{\infty} |A_{t(\lambda)}| \, d\lambda + \frac{5}{4} \int_{Q_x} |Q_x \cap 3A_{t(\lambda)}| \, d\lambda. \]

Since

\[ \lambda_0 + \int_{\lambda_0}^{\infty} |A_{t(\lambda)}| \, d\lambda = \sup_{|E| \leq 1} \int_E g \]

and

\[ \frac{1}{|Q_x|} \int_0^{\infty} |Q_x \cap 3A_{t(\lambda)}| \, d\lambda = \frac{1}{|Q_{x/3}|} \int_0^{\infty} |Q_{x/3} \cap A_{t(\lambda)}| \, d\lambda \]

\[ = \frac{1}{|Q_{x/3}|} \int_{Q_{x/3}} g \leq Mg(x/3), \]

this yields the desired conclusion.

We establish \((*)\) using elementary geometry. Calling \( \mu E \) the one-dimensional measure of \( E \cap L_t \), we first observe that \( (\mu_0 R) / |R| \leq (\mu_0 Q_x) / |Q_x| \).
To see this, we note that we can change one side of $R$ at a time until $Q_x$ is reached without decreasing $(\mu_0 R)/\|R\|$ at any stage. Next, note that for $|t|$ small the ratio $(\mu_0 R)/\|R\|$ is maximized by a square nearly the same as $Q_x$. Considering extreme cases shows that for $|3t|\leq |x_2 - x_1|$, we have $4(\mu_0 R)/\|R\| \leq 9(\mu_0 Q_x)/\|Q_x\|$ and, hence,

$$4\ |D_t \cap R|/\|R\| \leq 9\ |D_t \cap Q_x|/\|Q_x\|, \quad 0 < 3t \leq |x_2 - x_1|.$$  

For this last range of $t$, $|D_t \cap Q_x| \leq 8\ |D_{3t} \cap Q_x|$, while for all larger $t$ we have $Q_x \subset D_{3t}$. Consequently,

$$4\ |D_t \cap R|/\|R\| \leq 5\ |D_{3t} \cap Q_x|/\|Q_x\|, \quad 0 < t < \infty.$$  

For $x \in Q$ we have $Q_x \subset Q$ and $D_{3t} \cap Q = Q \cap 3A_t$; hence

$$4\ |A_t \cap R|/\|R\| \leq 4\ |D_t \cap R|/\|R\| \leq 5\ |D_{3t} \cap Q_x|/\|Q_x\| = 5\ |Q_x \cap 3A_t|/\|Q_x\|.$$  

For $x \notin Q$ and $|A_t| \leq 1$ the analysis is trivial. Considering extreme cases shows every $R$ meeting the complement of $Q$ satisfies $4\ |A_t \cap R|/\|R\| \leq 4t = |A_t|$, and the proof is complete.

REFERENCES


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