PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 4, August 1983

CONFORMAL INEQUIVALENCE OF ANNULI AND THE FIRST-ORDER THEORY OF SUBGROUPS OF PSL(2, R)

LEE A. RUBEL

ABSTRACT. An algebraic proof is given of the classical fact that two different concentric circular annuli $A(r)$ and $A(s)$ are conformally inequivalent, where $A(r) = \{z \in \mathbf{C}: 1 < |z| < r\}$. Indeed, it is shown that the covering groups of these annuli are not elementarily equivalent in the context of PSL(2, **R**). Considering the universal covering surface as U, the upper half-plane, the covering group of a bounded plane domain is naturally contained in PSL(2, **R**) as the group of covering transformations.

It is a classical fact of long standing [**NEH**, pp. 333–334; **RUD**, pp. 282–284] that two different concentric circular annuli $A(r)$ and $A(s)$ are not conformally equivalent, where $A(r) = \{z \in \mathbf{C}: 1 < |z| < r\}$. We give here an algebraic proof of this fact —indeed, we show that the covering groups of the two annuli, although isomorphic as abstract groups, are not even elementarily equivalent when taken in the context of PSL(2, **R**). This will require some explanation.

Given a bounded plane domain G, the universal covering surface S of G is conformally equivalent to the upper half-plane U. As usual, S is built up as a space of equivalence classes under homotopy of paths in G with some fixed base point $z_0 \in G$, and is endowed with the structure of a Riemann surface in a natural way. For convenience, we identify S with U. The group of conformal one: one maps of U onto U is PSL(2, **R**) (abbreviated as "PSL"), consisting of all transformations

$$z \mapsto \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbf{R},\ ad - bc = 1,$$

which we designate by the 2×2 matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ (really the equivalence classes of such matrices with

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}\right).$$

The covering group $\mathcal{G}$ of G is the group of all $h \in$ PSL that leave points in G invariant. This means that if $\varphi: U \to G$ is the universal covering map, then $\varphi \circ h = \varphi$. It is a classical fact that a region G is conformally equivalent to a region G' iff their covering groups $\mathcal{G}$ and $\mathcal{G}'$ are conjugate to one another, i.e. $\mathcal{G}' = p^{-1}\mathcal{G}p$ for some $p \in$ PSL. (They are anticonformally equivalent when p is chosen with determinant -1.) See [**NEV**, Chapter I] for details of some of these results.

Received by the editors July 27, 1982 and, in revised form, September 29, 1982. Presented to the Society at the AMS regional meeting in East Lansing, November 1982.
1980 *Mathematics Subject Classification*. Primary 03B10, 20G20, 30C20.

0002-9939/82/0000-1184/$02.50

Our first lemma is a result of Schreier and van der Waerden (see [**OME**, Theorem 5.6.5, p. 68]) that identifies the automorphisms of PSL(2, **R**).

LEMMA 1. *If* α *is an automorphism of* PSL(2, **R**) *onto itself then* α *is either an inner automorphism or an anti-inner automorphism* (*which means that* α *is the composition of an inner automorphism with multiplication by* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$).

We see that G and G' are Riemann-equivalent (i.e. either conformally or anticonformally equivalent) iff there is an automorphism of PSL onto itself that carries $\mathcal{G}$ onto $\mathcal{G}'$. It is therefore appropriate to study the structure (PSL, $\mathcal{G}$) which consists of PSL as a group together with a distinguished predicate that says when an element of PSL belongs to the subgroup. Informally, we may think of this structure as the group table of PSL printed in black, except that the elements of $\mathcal{G}$ are printed in red.

We take the final step of considering only the *first-order* theory of the structure (PSL, $\mathcal{G}$). This permits statements in the language of group theory, including the distinguished predicate, but, roughly speaking, these statements must have finite length, and must quantify only over *elements* of PSL, not over subsets, functions, or relations. Thus, to say that $\mathcal{G}$ is Abelian is first-order, but not to say that $\mathcal{G}$ is simple.

In a sense, then, we are considering a system of "first order conformal invariants" or "FOCI". But whereas the FOCI of [**BHR**] are related to the ring of holomorphic functions on $\mathcal{G}$, the present FOCI are group-theoretic in character.

An important thing to point out is that we are studying $\mathcal{G}$ and $\mathcal{G}'$ as they sit in PSL, and not just as abstract groups. Thus, we shall say such things as "$\mathcal{G}$ is elementarily equivalent to $\mathcal{G}'$ in the *context* of PSL" (written $\mathcal{G}$ e.e. $\mathcal{G}'$ (con PSL)) to mean that (PSL, $\mathcal{G}$) and (PSL, $\mathcal{G}'$) have exactly the same true first-order statements.

Let us now nail down these ideas to our annulus situation. For convenience of notation, it is handy to study the annulus $A(r^2)$ where $r > 1$. Here, the universal covering surface is helical, and can be realized as the horizontal strip $S(r^2) = \{z \in \mathbf{C}\colon 0 < \operatorname{Im} z < r^2\}$, where the covering map $\varphi\colon S(r^2) \to A(r^2)$ is $\varphi(z) = \exp(-iz)$. The covering group is the infinite cyclic group generated by $z \mapsto z + 2\pi$. Now $S(r^2)$ is mapped onto U via $\exp(z\pi/r^2)$ so that in our normalization, $\mathcal{G}(r^2)$ is the infinite cyclic subgroup of PSL generated by the dilation

$$\begin{pmatrix} \sqrt{2}\,\pi/r & 0 \\ 0 & r/\sqrt{2}\,\pi \end{pmatrix}.$$

(As pointed out by Joseph Rotman, the simple fact of conformal inequivalence of $A(r^2)$ and $A(S^2)$ is immediate at this point, for it is enough to show that $\mathcal{G}(r^2)$ and $\mathcal{G}(s^2)$ are not conjugate in PSL. But if they were, then their generators would have the same eigenvalues, which they do not.)

Thus, changing notation slightly, we are led to study the structure (PSL, $(r^2)^{\mathbf{Z}}$), where the subgroup $(r^2)^{\mathbf{Z}}$ is defined by

$$(r^2)^{\mathbf{Z}} = \left\{\begin{pmatrix} r^m & 0 \\ 0 & 1/r^m \end{pmatrix} : m = 0, \pm 1, \pm 2, \ldots\right\}.$$

This brings us to the main result of our paper.

THEOREM 1. *If $r \neq s$, then $(r^2)^{\mathbf{Z}}$ and $(s^2)^{\mathbf{Z}}$ are not elementarily equivalent in the context of* PSL(2, **R**).

In particular, this implies that two different concentric circular annuli $A(r)$ and $A(s)$ are not Riemann equivalent.

We regard this result as just one step in the larger program of studying FOCI from the point of view of elementary equivalence of covering groups in the context of PSL(2, **R**). For example, we might desire a first-order distinction between bounded plane domains with two holes and those with three holes. Now the covering group of a bounded plane domain with n holes is the free group on n generators, so we are close in spirit to a famous problem of Tarski to decide, say, whether the free group on two generators is elementarily equivalent to the free group on three generators. The distinction is that Tarski's problem concerns free-floating abstract groups, whereas our problem concerns certain concrete free groups firmly located in the context of PSL(2, **R**). We will not discuss here the many other ramifications of studying elementary equivalence of structures in the context of a larger fixed structure except to mention in passing that although the problem of elementary equivalence of abstract Abelian groups has been completely solved (see [**SMI**] and [**EKL**]), putting them "in context" opens up many new problems. For example, when are two Abelian subgroups of PSL(2, **R**) elementarily equivalent in the context of PSL(2, **R**)?

We acknowledge valuable discussions with C. Ward Henson, Carl Jockusch and John Walter.

PROOF OF THEOREM 1. In order not to get bogged down in notation, we will handle one case, showing that $(\pi^2)^{\mathbf{Z}}$ is not e.e.$(e^2)^{\mathbf{Z}}$ con PSL. The general case goes the same way. Now $9.869 \leqslant \pi^2 \leqslant 9.870$ and $9.869 \leqslant e^{2m} \leqslant 9.870$ is impossible for $m \in \mathbf{Z}$, as is easily verified.

OBSERVATION 1. Let

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \neq \mathrm{id}$$

be any dilation. The elements of PSL that commute with a are precisely the dilations.

This is easily proved by direct computation.

DEFINITION. *Consider* (PSL, $\mathcal{G}$) *and let d be an element of* PSL. *We say that d is a $\mathcal{G}$-dilation if d commutes with every element of $\mathcal{G}$.*

DEFINITION. *The elements* $\left(\begin{smallmatrix} 1 & \tau \\ 0 & 1 \end{smallmatrix}\right)$ *of* PSL *are called translations.*

DEFINITION. *The elements* $\left(\begin{smallmatrix} 1 & 0 \\ \tau & 1 \end{smallmatrix}\right)$ *of* PSL *are called antitranslations.*

DEFINITION. *An element of* PSL *is called a slide if it is either a translation or an antitranslation.*

OBSERVATION 2. An element t of PSL is a slide iff either $t = \mathrm{id}$ or t is not a dilation, and for all dilations D_1 and D_2 the elements $D_1^{-1}tD_1$ and $D_2^{-1}tD_2$ commute.

DEFINITION. *Consider* (PSL, $\mathcal{G}$) *and let t be an element of* PSL. *We say that t is a $\mathcal{G}$-slide if it satisfies the condition in Observation* 2 *with D_1 and D_2 being $\mathcal{G}$-dilations.*

REMARK. When $\mathcal{G}$ is a group of dilations, the $\mathcal{G}$-dilations are the ordinary dilations and $\mathcal{G}$-slides are the ordinary slides.

PROOF OF OBSERVATION 2 (SKETCH). First of all, if D is a dilation and t is a translation, then $D^{-1}tD$ is a translation, and all translations commute with each other. Similarly if t is an antitranslation. For the converse direction, we note that

$$\begin{pmatrix} 1/\alpha & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} = \begin{pmatrix} a & b/\alpha^2 \\ c\alpha^2 & d \end{pmatrix},$$

$$\begin{pmatrix} 1/\beta & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \beta & 0 \\ 0 & 1/\beta \end{pmatrix} = \begin{pmatrix} a & b/\beta^2 \\ c\beta^2 & d \end{pmatrix}.$$

These commute when

$$\begin{pmatrix} a^2 + bc\dfrac{\beta^2}{\alpha^2} & \dfrac{ab}{\beta^2} + \dfrac{bd}{\alpha^2} \\ ac\alpha^2 + cd\beta^2 & bc\dfrac{\alpha^2}{\beta^2} + d^2 \end{pmatrix} = \begin{pmatrix} a^2 + bc\dfrac{\alpha^2}{\beta^2} & \dfrac{ab}{\alpha^2} + \dfrac{bd}{\beta^2} \\ ac\beta^2 + cd\alpha^2 & bc\dfrac{\beta^2}{\alpha^2} + d^2 \end{pmatrix}.$$

We are supposing this to be true for all α, β. It follows that $bc = 0$ and that either $a = d$ or $b = 0$. The rest follows by a case analysis.

DEFINITION. *We say that two slides are of the same type if they are either both translations or both antitranslations.*

OBSERVATION 3. Two slides are of the same type iff they commute with each other.

PROOF. Trivial.

DEFINITION. *Two $\mathcal{G}$-slides are of the same type iff they commute with each other.*

DEFINITION. *Let s be a $\mathcal{G}$-slide. We put an order* $\leqslant$ mod s *on all $\mathcal{G}$-slides t of the same type by saying that* $t \geqslant 0$ mod s *if there exists a $\mathcal{G}$-dilation D so that $t = D^{-1}sD$.*

The motivation is that if

$$s = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \delta & 0 \\ 0 & 1/\delta \end{pmatrix},$$

then

$$D^{-1}sD = \begin{pmatrix} 1 & \alpha/\delta^2 \\ 0 & 1 \end{pmatrix},$$

and δ^2 is always positive. Similarly for antitranslations.

Now if t is a slide, then $t^{9.869}$ is well defined. It is the unique slide K of the same type as t such that $t^{9869} = K^{1000}$. Indeed, if $t = \left(\begin{smallmatrix} 1 & \tau \\ 0 & 1 \end{smallmatrix}\right)$ then

$$t^{9.869} = \begin{pmatrix} 1 & 9.869\tau \\ 0 & 1 \end{pmatrix}.$$

Similarly with $t^{9.870}$.

We may now complete our proof. Consider the following first-order sentence in (PSL, $\mathcal{G}$).

($\#$) There exists an element D of $\mathcal{G}$ and there exists a $\mathcal{G}$-slide $s \neq \mathrm{id}$ such that for all $\mathcal{G}$-slides t of the same type as s, with $t > 0 \bmod s$,

$$t^{9.869} \leqslant D^{-1}tD \leqslant t^{9.870} \quad \mathrm{mod}\ s.$$

This is true for $\mathcal{G} = (\pi^2)^{\mathbf{Z}}$, but false for $\mathcal{G} = (e^2)^{\mathbf{Z}}$. To see that is true for $(\pi^2)^{\mathbf{Z}}$, we take

$$D = \begin{pmatrix} 1/\pi & 0 \\ 0 & \pi \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \qquad \tau > 0.$$

Our assertion boils down to the two statements

$$\begin{pmatrix} 1 & (9.870 - \pi^2)\tau \\ 0 & 1 \end{pmatrix} \geqslant 0 \quad \mathrm{mod}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & (\pi^2 - 9.869)\tau \\ 0 & 1 \end{pmatrix} \geqslant 0 \quad \mathrm{mod}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which are clearly true if $\tau > 0$.

Similarly, it is true that ($\#$) is false for $\mathcal{G} = (e^2)^{\mathbf{Z}}$. Suppose D and S satisfy our conditions. Then

$$D = \begin{pmatrix} e^m & 0 \\ 0 & 1/e^m \end{pmatrix}, \qquad m \in \mathbf{Z},$$

and

$$s = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad s = \begin{pmatrix} 1 & 0 \\ \sigma & 1 \end{pmatrix}.$$

If ($\#$) were true, we would eventually arrive at $9.869 \leqslant e^{2m} \leqslant 9.870$, which is impossible. The proof is complete.

We conclude by remarking that the first-order theory of the structures (PSL, $\mathcal{G}$), where $\mathcal{G}$ ranges over the covering groups of bounded plane domains, can be interpreted as giving conformal invariants. It follows from **[BHR]** that there is a model of set theory (ZFC) in which this system is incomplete. We do not know if there is a model of set theory in which this new system is *complete*.

References

[BHR] Joseph Becker, C. Ward Henson and Lee A. Rubel, *First-order conformal invariants*, Ann. of Math. (2) **112** (1980), 123–178.

[EKL] Paul C. Eklof and Edward R. Fischer, *The elementary theory of Abelian groups*, Ann. Math. Logic **4** (1972), 115–171.

[NEH] Zeev Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952.

[NEV] R. Nevanlinna, *Eindeutige analytische Funktionen*, Grundlehren Math. Wiss., Bd. 66, Springer, Berlin, 1953.

[OME] O. T. O'Meara, *Lectures on linear groups*, CBMS Regional Conf. Ser. in Math., no. 22, Amer. Math. Soc., Providence, R.I., 1974.

[RUD] Walter Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.

[SMI] Wanda Szmielew, *Elementary properties of Abelian groups*, Fund. Math. **41** (1955), 203–271.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801