

WHEN IS THE SEMIGROUP RING PERFECT?

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ABSTRACT. A characterization of perfect semigroup rings $A[G]$ is given by means of the properties of the ring A and the semigroup G .

It was proved in [10] that for a ring with unity A and a group G the group ring $A[G]$ is perfect if and only if A is perfect and G is finite. Some results on perfectness of semigroup rings were obtained by Domanov [3]. He reduced the problem of describing perfect semigroup rings $A[G]$ to checking that certain semigroup algebras derived from $A[G]$ satisfy polynomial identities. Further, a characterization of such PI -algebras over a field of characteristic zero was found in [2]. However, the obtained results are difficult to formulate and refer to some exterior constructions obscuring an insight into the properties of the semigroup.

The purpose of this paper is to completely characterize perfect semigroup rings by means of the properties of the semigroup and the coefficient ring. Our approach is quite different from that of [3] and omits PI -methods. It works in arbitrary characteristic and the final result is a natural strengthening of the conditions for $A[G]$ to be semilocal [7].

In what follows A will be an associative ring, G —a semigroup. A is said to be right perfect if it is semilocal with its Jacobson radical $J(A)$ (right) T -nilpotent (cf. [4]). By $E(G)$ we shall mean the set of idempotents of G . If $e \in E(G)$, then we put $G_e = \{g \in eGe \mid g \text{ is invertible in } eGe\}$.

In the sequel the following well-known facts on T -nilpotence and perfectness will be useful.

LEMMA 1 (cf. [3, 4]). 1. *If G is a nil semigroup with d.c.c. on right principal ideals, then G is T -nilpotent.*

2. *If H is an ideal in G and G has d.c.c. on right principal ideals, then the semigroups H , G/H also have d.c.c. on right principal ideals.*

3. *If H is an ideal in G , then $A[G]$ is perfect if and only if so are the rings $A[H]$, $A[G/H]$.*

By arguments similar to those from [7, Theorem 1], one can easily get

PROPOSITION 1. *Let K be a field. Assume that $K[G]$ is perfect. Then the K -algebra $K[G]/J(K[G])$ is finite dimensional.*

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We will start with the following

LEMMA 2. *Let G be a periodic semigroup. Assume that $E(G) = E_1 \cup \dots \cup E_s$ with $efe \in G_e$ for any $e, f \in E_i$. Then there exists a chain of ideals $H_1 \subset H_2 \subset \dots \subset H_n = G$ such that $H_1, H_{i+1}/H_i, i > 1$, are nil or completely 0-simple.*

PROOF. We may assume that G has a zero element θ . Let H_1 be the largest nil ideal in G and $G_1 = G/H_1$. By hypothesis $G_1eG_1 = G_1fG_1$ for any $e, f \in E_i, i = 1, \dots, s$. Thus G_1 has a minimal ideal of the form $G_1eG_1, e \in E(G) \setminus \{\theta\}$. Since G_1eG_1 has no nonzero nil ideals, then it is easily verified to be a 0-simple semigroup. Moreover, any decreasing chain of idempotents in G has length $\leq s$, and so G_1eG_1 is completely 0-simple (cf. [1]). Hence, the required chain of ideals in G may be built by continuing the above procedure.

Let p be a prime number or 0. In the latter case, by a p -group we shall mean the trivial group. The elements $e, f \in E(G)$ will be said to be p -equivalent if for any $g \in G$ the following statement holds:

$ege \in G_e$ if and only if $efge, egfe \in G_e$ and if it is the case, then $ege, efge, egfe$ belong to the same coset of a normal p -subgroup in G_e .

THEOREM 1. *Let K be a field with $\text{char } K = p$. Then $K[G]$ is perfect if and only if*

1. G is periodic,
2. G has d.c.c. on right principal ideals,
3. G has no infinite subgroups,
4. $E(G) = \bigcup_{i=1}^s E_i$ for some disjoint subsets E_i of mutually p -equivalent idempotents.

PROOF. Assume first that $K[G]$ is perfect. Then G is periodic by [6]. Since the ring $K[G]$ has d.c.c. on right principal ideals [4], then G satisfies the condition 2. Let H be a subgroup in G . If $e = e^2 \in H$, then $K[eGe] = eK[G]e$ is perfect [4]. It is easy to check that $K[G_e]$ is a direct summand of the left $K[G_e]$ -module $K[eGe]$, hence it is perfect [8]. Thus, the group G_e is finite [10], and so $H \subset G_e$ is finite.

Since $J(K[G])$ is T -nilpotent, then by Proposition 1 $K[G]$ is a locally finite K -algebra. Thus, condition 4 follows from the fact that $K[G]$ is semilocal and from [7].

Assume now that G satisfies conditions 1–4. Then, by Lemma 2, we have a chain of ideals $H_1 \subset H_2 \subset \dots \subset H_n = G$ with $G_1 = H_1, G_i = H_i/H_{i-1}, i = 2, \dots, n$, being nil or completely 0-simple. Moreover, all G_i inherit the assumptions on G . In the former case G_i is T -nilpotent by Lemma 1 and hence $K[G_i]$ is perfect. If G_i is completely 0-simple, then $G_i \simeq M^0(X^0, I, \Delta, P)$ is a Rees matrix semigroup over a group with zero X^0 with sandwich matrix P [1]. By 2 X is a finite group which easily implies that G_i is locally finite. Now, it follows from [7] that $K[G_i]$ is semilocal. We will check that $J(K[G_i])$ is nilpotent. If $e \in E(G_i) \setminus \{\theta\}$, then $G_i e G_i = G_i$ and $J(K[G_i])^2 \subset K[G_i]J(K[G_i]) = K[G_i]eK[G_i]J(K[G_i]) \subset K[G_i]eJ(K[G_i])$. Now $eG_i e \simeq X^0$ [1], is finite. Thus $eJ(K[G_i])e = J(K[eG_i e])$ is nilpotent from which follows the nilpotency of $J(K[G_i])$.

Since $K[G_i]$ is perfect for any $i = 1, \dots, n$, it then follows from Lemma 1 that $K[G]$ is perfect.

The necessity of condition 2 was proved in [3]. The fact that $K[G]$ -perfect implies G -locally finite may be also deduced, in a different way, from [3].

Let us notice that, under the hypotheses of Theorem 1, there is a bound on cardinalities of subgroups in G . More precisely, for $e, f \in E_i$, let us consider the transformations $\varphi: eGe \rightarrow fGf$, $\psi: fGf \rightarrow eGe$ defined by $\varphi(x) = fxf$, $\psi(y) = eye$. It may be verified that $\varphi(G_e) \subset G_f$, $\psi(G_f) \subset G_e$. Since some power of $\varphi\psi$ is the identity transformation, then the groups G_e, G_f have the same number of elements. Moreover, if $\text{char } K = 0$, then φ is an isomorphism with $\varphi^{-1} = \psi$.

The following example shows that there is an essential difference between the conditions for $K[G]$ to be perfect in distinct characteristics (that is not the case for the class of group rings).

EXAMPLE. Let I be an infinite set and $G = M(X, I, I, P)$ be a completely simple Rees matrix semigroup over a finite nontrivial p -group X , $p > 0$. Then $K[G]$ is local for any field K of characteristic p [7]. Moreover, as in the proof of Theorem 1, $J(K[G])$ is nilpotent and hence $K[G]$ is right and left perfect. Let e be the unity in X . Choose $e \neq g \in G$ and put $P = (p_{ij})$, $p_{ii} = e$, $p_{ij} = g$ for $i, j \in I$, $i \neq j$. Then $(e)_{ii} \in E(G)$ and $(e)_{ii}(e)_{kk}(e)_{jj}(e)_{ii} = (g^3)_{ii} \neq (g^2)_{ii} = (e)_{ii}(e)_{kk}(e)_{ii}$ for distinct elements $i, j, k \in I$. Thus, the idempotents $(e)_{ii}, (e)_{jj}$ cannot be q -equivalent for $q \neq p$ and $L[G]$ is not semilocal for any field L with $\text{char } L \neq p$.

It is known that if a ring A is T -nilpotent, then for any sequence of finite subsets A_1, A_2, \dots in A there exists $n > 1$ such that $A_1 A_2 \cdots A_n = 0$ [9]. Thus $A[G]$ is T -nilpotent for any semigroup G . Now, for arbitrary A , $A[G]$ is perfect if and only if so is $A/J(A)[G]$. Hence, the case of an arbitrary coefficient ring may be easily derived from the following

THEOREM 2. *Let A be an algebra over a field K . Assume that A is not T -nilpotent. Then $A[G]$ is perfect if and only if so are the rings $A, K[G]$.*

PROOF. Since $A[G] \simeq A \otimes_K K[G]$, then the result follows from Proposition 1 and from [5, Theorem 1.4 and Theorem 2.5].

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