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HILBERT SPACES INDUCED BY HILBERT SPACE VALUED FUNCTIONS

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Dedicated to Professor Mitsuru Ozawa on his 60th birthday

ABSTRACT. Let E be an arbitrary set and $\mathcal{F}(E)$ a linear space composed of all complex valued functions on E. Let $\mathcal{H}$ be a (possibly finite-dimensional) Hilbert space with inner product $(\ ,\ )_{\mathcal{H}}$. Let $\mathbf{h}$: $E \to \mathcal{H}$ be a function and consider the linear mapping L from $\mathcal{H}$ into $\mathcal{F}(E)$ defined by $(\mathbf{F}, \mathbf{h}(p))_{\mathcal{H}}$. We let $\tilde{\mathcal{H}}$ denote the range of L. Then we assert that $\tilde{\mathcal{H}}$ becomes a Hilbert space with a reproducing kernel composed of functions on E, and, moreover, it is uniquely determined by the mapping L, in a sense. Furthermore, we investigate several fundamental properties for the mapping L and its inverse.

1. Introduction. The author **[5, 6]** developed a general theory of integral transforms of Hilbert spaces and investigated miscellaneous concrete integral transforms by a unified method. The situation is as follows:

Let dm denote a σ finite positive measure. Let $L_2(dm)$ denote a usual separable Hilbert space composed of dm integrable complex valued functions F on a dm measurable set T and with finite norms $\|F\|^2_{L_2(dm)} = \int_T |F(t)|^2\, dm(t)$. For an arbitrary set E and any fixed complex valued function $h(t, p)$ on $T \times E$ satisfying $h(t, p) \in L_2(dm)$ for any $p \in E$, we consider the integral transform of $F \in L_2(dm)$,

$$f(p) = \int_T F(t)\, \overline{h(t, p)}\, dm(t), \tag{1.1}$$

and we investigate this integral transform and its inverse. The basic method is based on the general theory of reproducing kernels using the direct integral theory established by Schwartz **[7]**.

In this paper we show that we can develop by elementary methods the general theory **[6]** without the direct integral theory and, at the same time, with a much more general situation.

Further extensions of this paper will be considered in connection with the recent research of R. E. Curto and S. Salinas **[4]**. The refereee gave the author this point of view.

Let E be an arbitrary set and $\mathcal{F}(E)$ a linear space compsed of all complex valued functions on E. Let $\mathcal{H}$ be a (possibly finite-dimensional) Hilbert space with inner

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product $(\, , \,)_{\mathcal{H}}$. Let $\mathbf{h}$: $E \to \mathcal{H}$ be a function. Then we consider the linear mapping L from $\mathcal{H}$ into $\mathcal{F}(E)$ defined by

$$f(p) = (L\mathbf{F})(p) = (\mathbf{F}, \mathbf{h}(p))_{\mathcal{H}}. \tag{1.2}$$

We let $\tilde{\mathcal{H}}$ denote the range of L. Then we show that $\tilde{\mathcal{H}}$ forms a Hilbert space admitting a reproducing kernel composed of functions on E, and, moreover, it is naturally induced from the mapping. Furthermore, we investigate several fundamental properties for the mapping L and its inverse.

2. Construction of the range $\tilde{\mathcal{H}}$ of L. We first introduce the inner product $(\, , \,)_{\tilde{\mathcal{H}}}$ in $\tilde{\mathcal{H}}$ defined by

$$\|f\|_{\tilde{\mathcal{H}}} = \inf\{\|F\|_{\mathcal{H}};\ f = L\mathbf{F}\}. \tag{2.1}$$

Then we obtain

THEOREM 2.1. $[\tilde{\mathbf{H}}, (\, , \,)_{\tilde{\mathcal{H}}}]$ *is a (possibly finite-dimensional) Hilbert space admitting the reproducing kernel* $K(p, q)$ *defined by*

$$K(p, q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}}. \tag{2.2}$$

Moreover, L *is an isometry between* $\mathcal{H}$ *and* $\tilde{\mathcal{H}}$ *if and only if* $\{\mathbf{h}(p);\ p \in E\}$ *is complete in* $\mathcal{H}$.

PROOF. From (1.2), null(L) is a closed subspace in $\mathcal{H}$. Hence, for $f = L\mathbf{F}$, we have

$$\|f\|_{\tilde{\mathcal{H}}} = \inf\{\|\mathbf{F} - \mathbf{G}\|_{\mathcal{H}};\ \mathbf{G} \in \mathrm{null}(L)\} = \|P_{\mathcal{G}}\mathbf{F}\|_{\mathcal{H}}. \tag{2.3}$$

Here, $P_{\mathcal{G}}$ is an orthogonal projection from $\mathcal{H}$ onto $\mathcal{G} = \mathcal{H} \ominus \mathrm{null}(L)$. When we restrict L on $\mathcal{G}$, then $L|_{\mathcal{G}}$ is an isometry between $[\mathcal{G}, (\, , \,)_{\mathcal{H}}]$ and $[\tilde{\mathcal{H}}, (\, , \,)_{\tilde{\mathcal{H}}}]$, which implies that $[\tilde{\mathcal{H}}, (\, , \,)_{\tilde{\mathcal{H}}}]$ is a Hilbert space.

Next, we note that when $\mathbf{F} \in \mathrm{null}(L)$, then

$$(\mathbf{F}, \mathbf{h}(p))_{\mathcal{H}} = 0 \quad \text{for all } p \in E. \tag{2.4}$$

Hence, for any $q \in E$, $\mathbf{h}(q) \in \mathcal{G}$. From this fact we have, for any $f = L\mathbf{F}$,

$$(f, K(\cdot, q))_{\tilde{\mathcal{H}}} = (L\mathbf{F}, L\mathbf{h}(q))_{\tilde{\mathcal{H}}} = (P_{\mathcal{G}}\mathbf{F}, P_{\mathcal{G}}\mathbf{h}(q))_{\mathcal{H}} = (\mathbf{F}, \mathbf{h}(q))_{\mathcal{H}} = f(q), \tag{2.5}$$

which implies that $K(p, q)$ is the reproducing kernel for $\tilde{\mathcal{H}}$.

In the last, we note that when $\{\mathbf{h}(p);\ p \in E\}$ is complete in $\mathcal{H}$, then $\mathcal{G} = \mathcal{H}$ so we have, for $f = L\mathbf{F}$,

$$\|f\|_{\tilde{\mathcal{H}}} = \|\mathbf{F}\|_{\mathcal{H}} \tag{2.6}$$

and vice versa. We thus complete the proof of Theorem 2.1.

3. Direct construction of the space $\tilde{\mathcal{H}}$. In Theorem 2.1 the norm in $\tilde{\mathcal{H}}$ is given in terms of the norm in $\mathcal{H}$ by making use of the mapping L. Here we note that we can construct $\tilde{\mathcal{H}}$ directly, in principle. This point of view is essentially important in dealing with concrete integral transforms. See **[5, 6]**. In order to show this fact, we note that the function $K(p, q)$ on $E \times E$ is positive definite on E; that is,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} K(p_i, p_j) \geqslant 0$$

for any finite set $\{p_i\}$ of E and for any complex numbers $\{\alpha_i\}$, as we see directly from (2.2). Then the theory of Moore and Aronszajn (see **[1, 2]**) implies that for this $K(p, q)$ there exists a *uniquely determined* Hilbert space composed of functions on E admitting $K(p, q)$ as the reproducing kernel. This space is just $\tilde{\mathcal{H}}$ stated.

We assume in the sequel that for $K(p, q)$, the space $\tilde{\mathcal{H}}$ can be realized in this sense. For miscellaneous concrete examples, see **[6]**. Then we have, of course, the following fundamental inequality for L:

THEOREM 3.1. *For the linear mapping*

$$f(p) = (\mathbf{F}, \mathbf{h}(p))_{\mathcal{H}}, \tag{1.2}$$

we obtain the inequality

$$\|f\|_{\tilde{\mathcal{H}}} \leq \|\mathbf{F}\|_{\mathcal{H}}. \tag{3.1}$$

When $\{\mathbf{h}(p);\ p \in E\}$ *is complete in* $\mathcal{H}$, *then equality in* (3.1) *always holds.*

Let $\{\mathbf{F}_j\}$ be an orthonormal basis for $\mathcal{H}$. Then

$$\mathbf{h}(p) = \sum_j (\mathbf{h}(p), \mathbf{F}_j)\mathbf{F}_j = \sum_j \overline{f_j(p)}\,\mathbf{F}_j \quad \text{and} \quad \overline{\mathbf{h}(p)} = \sum_j f_j(p)\mathbf{F}_j.$$

Thus $\bar{\mathbf{h}} = \sum_j f_j(\cdot)\mathbf{F}_j$. We define

$$(f, \mathbf{h})_{\tilde{\mathcal{H}}} = \sum_j (f, f_j)_{\tilde{\mathcal{H}}}\mathbf{F}_j.$$

Then we obtain

THEOREM 3.2. *We assume that for* $f \in \tilde{\mathcal{H}}$,

$$(f, \mathbf{h})_{\tilde{\mathcal{H}}} \in \mathcal{H} \tag{3.2}$$

and

$$(f, (\mathbf{h}(q), \mathbf{h}(\cdot))_{\mathcal{H}})_{\tilde{\mathcal{H}}} = ((f, \mathbf{h})_{\tilde{\mathcal{H}}}, \mathbf{h}(q))_{\mathcal{H}} \quad \textit{for all } q \in E. \tag{3.3}$$

Then we obtain the inequality

$$\|f\|_{\tilde{\mathcal{H}}} \leq \|(f, \mathbf{h})_{\tilde{\mathcal{H}}}\|_{\mathcal{H}}. \tag{3.4}$$

When $\{\mathbf{h}(p);\ p \in E\}$ *is complete in* $\mathcal{H}$, *then equality always holds in* (3.4).

PROOF. We note that for any $q \in E$,

$$f(p) = (f(\cdot), K(\cdot, p))_{\tilde{\mathcal{H}}} = (f(\cdot), (\mathbf{h}(p), \mathbf{h}(\cdot))_{\mathcal{H}})_{\tilde{\mathcal{H}}} = ((f, \mathbf{h})_{\tilde{\mathcal{H}}}, \mathbf{h}(p))_{\mathcal{H}}. \tag{3.5}$$

This shows that $f = L(f, \mathbf{h})_{\tilde{\mathcal{H}}} = L\mathbf{F}$, so by the definition of $\|\cdot\|_{\tilde{\mathcal{H}}}$ (see Theorem 3.1) we have $\|f\|_{\tilde{\mathcal{H}}} \leq \|(f, \mathbf{h})_{\tilde{\mathcal{H}}}\|_{\mathcal{H}}$.

4. The inverse of L. We consider the inverse of L. Following Theorem 2.1, we need the assumption that $\{\mathbf{h}(p);\ p \in E\}$ is complete in $\mathcal{H}$. Then from the identity (3.5), we obtain

THEOREM 4.1. *We assume* (3.2) *and* (3.3) *are valid, and* $\{\mathbf{h}(p);\ p \in E\}$ *is complete in* $\mathcal{H}$. *Then for* $f(p) = (\mathbf{F}, \mathbf{h}(p))_{\mathcal{H}}$, *we obtain its inverse*

$$\mathbf{F} = (f, \mathbf{h})_{\tilde{\mathcal{H}}}. \tag{4.1}$$

When $\{\mathbf{h}(p);\ p \in E\}$ is not complete in $\mathcal{H}$, for f we let $\mathbf{F}^*$ be the vector satisfying

$$\|f\|_{\tilde{\mathcal{H}}} = \|\mathbf{F}^*\|_{\mathcal{H}} \tag{4.2}$$

in (3.1). Then $\mathbf{F}^*$ is, of course, uniquely determined in $\mathcal{H}$. As to this vector $\mathbf{F}^*$, we obtain directly

THEOREM 4.2. *We assume that* (3.2) *and* (3.3) *are valid and, further,*

$$(\mathbf{F}_0, (f, \mathbf{h})_{\tilde{\mathcal{H}}})_{\mathcal{H}} = ((\mathbf{F}_0, \mathbf{h})_{\mathcal{H}}, f)_{\tilde{\mathcal{H}}} \quad \textit{for all } \mathbf{F}_0 \in \operatorname{null}(L). \tag{4.3}$$

Then we have

$$\mathbf{F}^* = (f, \mathbf{h})_{\tilde{\mathcal{H}}}. \tag{4.4}$$

As we see from concrete examples, conditions (3.2) and (3.3) are, in general, strong. For a more general and weak inverse formula in the case that the norm of $\tilde{\mathcal{H}}$ is realized by a positive measure, see **[5, 6]**.

5. Generating functions and Hilbert space valued functions. We note that the starting point in our theory is, in fact, the identity (2.2). We now recall that there exists a general method which gives such an identity from a concrete Hilbert space $\tilde{\mathcal{H}}$ with the reproducing kernel $K(p, q)$ and an isometry mapping $\tilde{L}$ between $\tilde{\mathcal{H}}$ and $\mathcal{H}$, conversely. See Shapiro and Shields **[8]** and Burbea **[3]** for many concrete examples. The image of $K(\cdot, q)$ under this isometry $\tilde{L}$ is denoted by

$$\mathbf{g}_{\tilde{L}}(q) = \tilde{L}K(\cdot, q), \tag{5.1}$$

which is called "the generating vector of $\tilde{L}$". Then we have

(5.2)

$$K(p, q) = (K(\cdot, q), K(\cdot, p))_{\tilde{\mathcal{H}}} = (\tilde{L}K(\cdot, q), \tilde{L}K(\cdot, p))_{\mathcal{H}} = (\mathbf{g}_{\tilde{L}}(q), \mathbf{g}_{\tilde{L}}(p))_{\mathcal{H}},$$

which is of type (2.2). From this identity and our theory, we obtain

THEOREM 5.1. *For the linear mapping*

$$f(p) = (\mathbf{F}, \mathbf{g}_{\tilde{L}}(p))_{\mathcal{H}}, \qquad \mathbf{F} \in \mathcal{H}, \tag{5.3}$$

we have the identity

$$\|f\|_{\tilde{\mathcal{H}}} = \|\mathbf{F}\|_{\mathcal{H}}. \tag{5.4}$$

Further, the mapping (5.3) *gives the isometry $\tilde{L}$ and the family of vectors $\{\mathbf{g}_{\tilde{L}}(p);\ p \in E\}$ is complete in $\mathcal{H}$.*

Moreover, when for $f \in \tilde{\mathcal{H}}$,

$$(f, \mathbf{g}_{\tilde{L}})_{\tilde{\mathcal{H}}} \in \mathcal{H} \tag{5.5}$$

and

$$(f, (\mathbf{g}_{\tilde{L}}(q), \mathbf{g}_{\tilde{L}})_{\mathcal{H}})_{\tilde{\mathcal{H}}} = ((f, \mathbf{g}_{\tilde{L}})_{\tilde{\mathcal{H}}}, \mathbf{g}_{\tilde{L}}(q))_{\mathcal{H}} \quad \textit{for all } q \in E \tag{5.6}$$

are valid, then we obtain the inverse of (5.3),

$$\mathbf{F} = (f, \mathbf{g}_{\tilde{L}})_{\tilde{\mathcal{H}}}. \tag{5.7}$$

PROOF. We note that in (5.3), $\tilde{L}f = \mathbf{F}$ for any $\mathbf{F} \in \mathcal{H}$. Indeed, since $\tilde{L}$ is an isometry, for $\tilde{L}\tilde{f} = \mathbf{F}$, we have

$$(5.8) \quad f(p) = (\mathbf{F}, \mathbf{g}_{\tilde{L}}(p))_{\mathcal{H}} = (\tilde{L}\tilde{f}, \tilde{L}K(\cdot, p))_{\mathcal{H}} = (\tilde{f}(\cdot), K(\cdot, p))_{\tilde{\mathcal{H}}} = \tilde{f}(p).$$

Hence, $f \equiv \tilde{f}$ and we see that (5.3) gives the isometry $\tilde{L}$. Of course, we have (5.4). Further, from (5.8), we see that $\{\mathbf{g}_{\tilde{L}}(p); p \in E\}$ is complete in $\mathcal{H}$. Moreover, from the argument in Theorem 3.2, we have the inverse formula (5.7).

For miscellaneous concrete examples, see [**5**, **6**].

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