HILBERT SPACES INDUCED
BY HILBERT SPACE VALUED FUNCTIONS

SABUROU SAITO

Dedicated to Professor Mitsuru Ozawa on his 60th birthday

Abstract. Let $E$ be an arbitrary set and $\mathcal{F}(E)$ a linear space composed of all complex valued functions on $E$. Let $\mathcal{X}$ be a (possibly finite-dimensional) Hilbert space with inner product $(\cdot, \cdot)_\mathcal{X}$. Let $h : E \to \mathcal{X}$ be a function and consider the linear mapping $L$ from $\mathcal{X}$ into $\mathcal{F}(E)$ defined by $(F, h(p))_\mathcal{X}$. We let $\mathcal{K}$ denote the range of $L$. Then we assert that $\mathcal{K}$ becomes a Hilbert space with a reproducing kernel composed of functions on $E$, and, moreover, it is uniquely determined by the mapping $L$, in a sense. Furthermore, we investigate several fundamental properties for the mapping $L$ and its inverse.

1. Introduction. The author [5, 6] developed a general theory of integral transforms of Hilbert spaces and investigated miscellaneous concrete integral transforms by a unified method. The situation is as follows:

Let $dm$ denote a $\sigma$ finite positive measure. Let $L_2(dm)$ denote a usual separable Hilbert space composed of $dm$ integrable complex valued functions $F$ on a $dm$ measurable set $T$ and with finite norms $\|F\|_{L_2(dm)}^2 = \int_T |F(t)|^2 \, dm(t)$. For an arbitrary set $E$ and any fixed complex valued function $h(t, p)$ on $T \times E$ satisfying $h(t, p) \in L_2(dm)$ for any $p \in E$, we consider the integral transform of $F \in L_2(dm)$,

$$f(p) = \int_T F(t) \overline{h(t, p)} \, dm(t),$$

and we investigate this integral transform and its inverse. The basic method is based on the general theory of reproducing kernels using the direct integral theory established by Schwartz [7].

In this paper we show that we can develop by elementary methods the general theory [6] without the direct integral theory and, at the same time, with a much more general situation.

Further extensions of this paper will be considered in connection with the recent research of R. E. Curto and S. Salinas [4]. The referee gave the author this point of view.

Let $E$ be an arbitrary set and $\mathcal{F}(E)$ a linear space composed of all complex valued functions on $E$. Let $\mathcal{K}$ be a (possibly finite-dimensional) Hilbert space with inner product $(\cdot, \cdot)_{\mathcal{K}}$.
product \((\ , \ )_\mathcal{H}\). Let \(h: E \to \mathcal{H}\) be a function. Then we consider the linear mapping \(L\) from \(\mathcal{H}\) into \(\mathcal{F}(E)\) defined by
\[
(1.2) \quad f(p) = (LF)(p) = (F, h(p))_\mathcal{H}.
\]
We let \(\bar{\mathcal{H}}\) denote the range of \(L\). Then we show that \(\bar{\mathcal{H}}\) forms a Hilbert space admitting a reproducing kernel composed of functions on \(E\), and, moreover, it is naturally induced from the mapping. Furthermore, we investigate several fundamental properties for the mapping \(L\) and its inverse.

2. Construction of the range \(\bar{\mathcal{H}}\) of \(L\). We first introduce the inner product \((\ , \ )_{\bar{\mathcal{H}}}\) in \(\bar{\mathcal{H}}\) defined by
\[
(2.1) \quad \|f\|_{\bar{\mathcal{H}}} = \inf\{\|F\|_{\mathcal{H}}; f = LF\}.
\]
Then we obtain

**Theorem 2.1.** \([\bar{\mathcal{H}}, (\ , \ )_{\bar{\mathcal{H}}}]\) is a (possibly finite-dimensional) Hilbert space admitting the reproducing kernel \(K(p, q)\) defined by
\[
(2.2) \quad K(p, q) = (h(q), h(p))_{\mathcal{H}}.
\]
Moreover, \(L\) is an isometry between \(\mathcal{H}\) and \(\bar{\mathcal{H}}\) if and only if \(\{h(p); p \in E\}\) is complete in \(\mathcal{H}\).

**Proof.** From (1.2), \(\text{null}(L)\) is a closed subspace in \(\mathcal{H}\). Hence, for \(f = LF\), we have
\[
(2.3) \quad \|f\|_{\bar{\mathcal{H}}} = \inf\{\|F - G\|_{\mathcal{H}}; G \in \text{null}(L)\} = \|P_\mathcal{H}F\|_{\mathcal{H}}.
\]
Here, \(P_\mathcal{H}\) is an orthogonal projection from \(\mathcal{H}\) onto \(\mathcal{H} = \mathcal{H} \oplus \text{null}(L)\). When we restrict \(L\) on \(\mathcal{H}\), then \(L|_{\mathcal{H}}\) is an isometry between \([\mathcal{H}, (\ , \ )_\mathcal{H}]\) and \([\bar{\mathcal{H}}, (\ , \ )_{\bar{\mathcal{H}}}]\), which implies that \([\bar{\mathcal{H}}, (\ , \ )_{\bar{\mathcal{H}}}]\) is a Hilbert space.

Next, we note that when \(F \in \text{null}(L)\), then
\[
(2.4) \quad (F, h(p))_{\mathcal{H}} = 0 \quad \text{for all} \, p \in E.
\]
Hence, for any \(q \in E\), \(h(q) \in \mathcal{H}\). From this fact we have, for any \(f = LF\),
\[
(2.5) \quad (f, K(\cdot, q))_{\bar{\mathcal{H}}} = (LF, Lh(q))_{\mathcal{H}} = (P_\mathcal{H}F, P_\mathcal{H}h(q))_{\mathcal{H}} = (F, h(q))_{\mathcal{H}} = f(q),
\]
which implies that \(K(p, q)\) is the reproducing kernel for \(\bar{\mathcal{H}}\).

In the last, we note that when \(\{h(p); p \in E\}\) is complete in \(\mathcal{H}\), then \(\mathcal{H} = \mathcal{H}\) so we have, for \(f = LF\),
\[
(2.6) \quad \|f\|_{\bar{\mathcal{H}}} = \|F\|_{\mathcal{H}}
\]
and vice versa. We thus complete the proof of Theorem 2.1.

3. Direct construction of the space \(\bar{\mathcal{H}}\). In Theorem 2.1 the norm in \(\bar{\mathcal{H}}\) is given in terms of the norm in \(\mathcal{H}\) by making use of the mapping \(L\). Here we note that we can construct \(\bar{\mathcal{H}}\) directly, in principle. This point of view is essentially important in dealing with concrete integral transforms. See [5, 6]. In order to show this fact, we note that the function \(K(p, q)\) on \(E \times E\) is positive definite on \(E\); that is,
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \overline{\alpha_j} K(p_i, p_j) \geq 0
\]
for any finite set \( \{ p_j \} \) of \( E \) and for any complex numbers \( \{ \alpha_j \} \), as we see directly from (2.2). Then the theory of Moore and Aronszajn (see [1, 2]) implies that for this \( K(p, q) \) there exists a uniquely determined Hilbert space composed of functions on \( E \) admitting \( K(p, q) \) as the reproducing kernel. This space is just \( \mathcal{K} \) stated.

We assume in the sequel that for \( K(p, q) \), the space \( \mathcal{K} \) can be realized in this sense. For miscellaneous concrete examples, see [6]. Then we have, of course, the following fundamental inequality for \( L \):

\[ \text{Theorem 3.1. For the linear mapping} \]
\[ (1.2) \quad f(p) = (F, h(p))_{\mathcal{K}}, \]

we obtain the inequality
\[ (3.1) \quad \| f \|_{\mathcal{K}} \leq \| F \|_{\mathcal{K}}. \]

When \( \{ h(p); p \in E \} \) is complete in \( \mathcal{K} \), then equality in (3.1) always holds.

Let \( \{ F_j \} \) be an orthonormal basis for \( \mathcal{K} \). Then
\[ h(p) = \sum_j (h(p), F_j) F_j = \sum_j f_j(p) F_j \quad \text{and} \quad \overline{h(p)} = \sum_j f_j(p) F_j. \]

Thus \( \overline{h} = \sum_j f_j(\cdot)F_j \). We define
\[ (f, h)_{\mathcal{K}} = \sum_j (f, f_j)_{\mathcal{K}} F_j. \]

Then we obtain

\[ \text{Theorem 3.2. We assume that for } f \in \mathcal{K}, \]
\[ (3.2) \quad (f, h)_{\mathcal{K}} \in \mathcal{K} \]

and
\[ (3.3) \quad (f, (h(q), h(\cdot)))_{\mathcal{K}} = ((f, h)_{\mathcal{K}}, h(q))_{\mathcal{K}} \quad \text{for all } q \in E. \]

Then we obtain the inequality
\[ (3.4) \quad \| f \|_{\mathcal{K}} \leq \| (f, h)_{\mathcal{K}} \|_{\mathcal{K}}. \]

When \( \{ h(p); p \in E \} \) is complete in \( \mathcal{K} \), then equality always holds in (3.4).

\[ \text{Proof. We note that for any } q \in E, \]
\[ (3.5) \quad f(p) = (f(\cdot), K(\cdot, p))_{\mathcal{K}} = (f(\cdot), (h(p), h(\cdot)))_{\mathcal{K}} = ((f, h)_{\mathcal{K}}, h(p))_{\mathcal{K}}. \]

This shows that \( f = L(f, h)_{\mathcal{K}} = LF \), so by the definition of \( \| \cdot \|_{\mathcal{K}} \) (see Theorem 3.1) we have \( \| f \|_{\mathcal{K}} \leq \| (f, h)_{\mathcal{K}} \|_{\mathcal{K}}. \)

4. The inverse of \( L \). We consider the inverse of \( L \). Following Theorem 2.1, we need the assumption that \( \{ h(p); p \in E \} \) is complete in \( \mathcal{K} \). Then from the identity (3.5), we obtain

\[ \text{Theorem 4.1. We assume (3.2) and (3.3) are valid, and } \{ h(p); p \in E \} \text{ is complete in } \mathcal{K}. \text{ Then for } f(p) = (F, h(p))_{\mathcal{K}}, \text{ we obtain its inverse} \]
\[ (4.1) \quad F = (f, h)_{\mathcal{K}}. \]
When \( \{h(p); p \in E\} \) is not complete in \( \mathcal{K} \), for \( f \) we let \( F^* \) be the vector satisfying

\[
\|f\|_{\mathcal{K}} = \|F^*\|_{\mathcal{K}}
\]

in (3.1). Then \( F^* \) is, of course, uniquely determined in \( \mathcal{K} \). As to this vector \( F^* \), we obtain directly:

**Theorem 4.2.** We assume that (3.2) and (3.3) are valid and, further,

\[
(F_0, (f, h)_{\mathcal{K}})_{\mathcal{K}} = ((F_0, h)_{\mathcal{K}}, f)_{\mathcal{K}}
\]

for all \( F_0 \in \text{null}(L) \).

Then we have

\[
F^* = (f, h)_{\mathcal{K}}.
\]

As we see from concrete examples, conditions (3.2) and (3.3) are, in general, strong. For a more general and weak inverse formula in the case that the norm of \( \mathcal{K} \) is realized by a positive measure, see [5,6].

5. Generating functions and Hilbert space valued functions. We note that the starting point in our theory is, in fact, the identity (2.2). We now recall that there exists a general method which gives such an identity from a concrete Hilbert space \( \mathcal{K} \) with the reproducing kernel \( K(p, q) \) and an isometry mapping \( \hat{L} \) between \( \mathcal{K} \) and \( \mathcal{K} \), conversely. See Shapiro and Shields [8] and Burbea [3] for many concrete examples. The image of \( K(\cdot, q) \) under this isometry \( \hat{L} \) is denoted by

\[
g_{\hat{L}}(q) = \hat{L}K(\cdot, q),
\]

which is called “the generating vector of \( \hat{L} \)”. Then we have

\[
K(p, q) = (K(\cdot, q), K(\cdot, p))_{\mathcal{K}} = (\hat{L}K(\cdot, q), \hat{L}K(\cdot, p))_{\mathcal{K}} = (g_{\hat{L}}(q), g_{\hat{L}}(p))_{\mathcal{K}},
\]

which is of type (2.2). From this identity and our theory, we obtain

**Theorem 5.1.** For the linear mapping

\[
f(p) = (F, g_{\hat{L}}(p))_{\mathcal{K}}, \quad F \in \mathcal{K},
\]

we have the identity

\[
\|f\|_{\mathcal{K}} = \|F\|_{\mathcal{K}}.
\]

Further, the mapping (5.3) gives the isometry \( \hat{L} \) and the family of vectors \( \{g_{\hat{L}}(p); p \in E\} \) is complete in \( \mathcal{K} \).

Moreover, when for \( f \in \mathcal{K} \),

\[
(f, g_{\hat{L}})_{\mathcal{K}} \in \mathcal{K}
\]

and

\[
(f, (g_{\hat{L}}(q), g_{\hat{L}})_{\mathcal{K}})_{\mathcal{K}} = ((f, g_{\hat{L}})_{\mathcal{K}}, g_{\hat{L}}(q))_{\mathcal{K}}
\]

for all \( q \in E \), are valid, then we obtain the inverse of (5.3),

\[
F = (f, g_{\hat{L}})_{\mathcal{K}}.
\]
Proof. We note that in (5.3), \( Lf = F \) for any \( F \in \mathcal{H} \). Indeed, since \( L \) is an isometry, for \( \tilde{L}f = F \), we have

\[
(5.8) \quad f(p) = (F, g\tilde{L}(p))_{\mathcal{H}} = (\tilde{L}f, \tilde{L}K(\cdot, p))_{\mathcal{H}} = (\tilde{f}(\cdot), K(\cdot, p))_{\mathcal{H}} = \tilde{f}(p).
\]

Hence, \( f \equiv \tilde{f} \) and we see that (5.3) gives the isometry \( \tilde{L} \). Of course, we have (5.4). Further, from (5.8), we see that \( \{g\tilde{L}(p); \ p \in E\} \) is complete in \( \mathcal{H} \). Moreover, from the argument in Theorem 3.2, we have the inverse formula (5.7).

For miscellaneous concrete examples, see [5, 6].

Acknowledgments. The author wishes to thank Professors T. Ando, F. Beatrous, Jr., and the referee for their valuable advice and comments.

References