PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 1, September 1983

FACTORIZATION IN CODIMENSION TWO IDEALS OF GROUP ALGEBRAS

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ABSTRACT. Let G be a finitely generated group and I be a closed, two-sided ideal with codimension two in $L^1(G)$. Then the linear span of the set of all products in I is equal to I.

Let A be a complex algebra. For $I, J \subseteq A$, define $IJ = \{\sum_{k=1}^{n} a_k b_k \mid n \in \mathbf{N}; a_k \in I, b_k \in J, k = 1, \ldots, n\}$, and abbreviate II as I^2. It is clear that if I is an ideal in A, then $I^2 \subseteq I$ (by *ideal* will always be meant a two-sided ideal). The ideal I is said to be *idempotent* if $I^2 = I$.

Now suppose that A is also a Banach algebra. Then a question which arises in connection with automatic continuity problems for A is whether I is idempotent whenever I is a closed finite-codimensional ideal in A. For example, see [**2**, §6] and [**3**].

This question is particularly interesting when A is the group algebra, $L^1(G)$, of a locally compact group G, as there are several classes of groups such that every finite-codimensional ideal in $L^1(G)$ is idempotent. For example, it is an immediate consequence of Theorem 2 in [**5**] that, if G is amenable, then every closed, finite-codimensional ideal in $L^1(G)$ has a bounded approximate identity. Hence, by Cohen's factorization theorem [**1**, Theorem 11.10], every such ideal is idempotent. It is also shown in [**8**] that, if G is connected, then every closed, finite-codimensional ideal in $L^1(G)$ is idempotent. Furthermore, in [**6**] it is shown that closed ideals with codimension one in $L^1(G)$ are idempotent for every G. Every group algebra has at least one codimension one ideal, namely the *augmentation ideal* $I_0(G) = \{f \in L^1(G) \mid \int_G f\,dx = 0\}$.

This paper is concerned with ideals with codimension two in $L^1(G)$. The main theorem deals with finitely generated groups. Since a finitely generated group G is countable, Haar measure on G will be discrete. Hence we may normalize it to be counting measure and take $L^1(G)$ to be the set of functions f on G with

$$\|f\| = \sum_{x \in G} |f(x)| < \infty.$$

The function which takes the value one at x and is zero elsewhere will be denoted by $\bar{x}$. If 1 is the identity element of G, then $\bar{1}$ is a multiplicative identity for $L^1(G)$.

Received by the editors September 9, 1982.
1980 *Mathematics Subject Classification.* Primary 43A20; Secondary 46H10.

THEOREM 1. *Let G be a finitely generated group and I be a closed, two-sided ideal with codimension two in $L^1(G)$. Then I is idempotent.*

PROOF. Consider $L^1(G)/I$. It is a two-dimensional complex algebra with unit, and is semisimple by [**3**, Lemma 3.1]. (Alternatively, $L^1(G)/I$ may be shown to be semisimple in the following way. Let $\rho: L^1(G) \to L^1(G)/I$ be the quotient homomorphism and R be the radical of $L^1(G)/I$. Then if $R \neq (0)$, $\rho^{-1}(R)$ is a non-idempotent codimension one ideal in $L^1(G)$, which contradicts the theorem of [**6**].) Hence $L^1(G)/I$ is isomorphic to $\mathbf{C} \oplus \mathbf{C}$ and so there are multiplicative linear functionals ϕ_0 and ϕ_1 on $L^1(G)$ such that $I = \ker \phi_0 \cap \ker \phi_1$.

Let χ_0 and χ_1 be the characters on G such that

$$\phi_j(f) = \sum_{x \in G} f(x)\chi_j(x) \qquad (f \in L^1(G), j = 0, 1)$$

(see [**4**, Corollary 23.7]). Define an automorphism T of $L^1(G)$ by

$$(Tf)(x) = f(x)\chi_0(x) \qquad (x \in G, f \in L^1(G)).$$

Then $T(I)$ is a codimension two ideal in $L^1(G)$ and in order to show that I is idempotent it will suffice to show that $T(I)$ is idempotent. Hence, replacing I by $T(I)$ and ϕ_j by $\phi_j \circ T^{-1}$, we may suppose that χ_0 is the trivial character and $\ker \phi_0$ is the augmentation ideal of $L^1(G)$.

Now suppose that G is generated by n elements $y_1, y_2, \ldots, y_n$ and let $\mathbf{F}_n$ be the free group on n generators $x_1, x_2, \ldots, x_n$. Then there is a surjective group homomorphism $q: \mathbf{F}_n \to G$ defined by

$$q(x_i) = y_i, \qquad i = 1, 2, \ldots, n,$$

and a surjective algebra homomorphism $Q: L^1(\mathbf{F}_n) \to L^1(G)$ defined by

$$(Qf)(y) = \sum_{x \in q^{-1}(y)} f(x) \qquad (y \in G, f \in L^1(\mathbf{F}_n)).$$

It is clear that $Q^{-1}(I)$ is a codimension two ideal in $L^1(\mathbf{F}_n)$ and that I is idempotent if $Q^{-1}(I)$ is. Hence it will suffice to prove the theorem in the case when $G = \mathbf{F}_n$.

If χ_1 was the trivial character, then we would have that $\phi_0 = \phi_1$ and I would have codimension one. Thus χ_1 is not trivial and we may suppose that $\chi_1(x_1) \neq 1$. Now if $\chi_1(x_i) = 1$ for some i, then $\chi_1(x_1x_i) \neq 1$ and $\{x_1, \ldots, x_{i-1}, x_1x_i, x_{i+1}, \ldots, x_n\}$ still generates $\mathbf{F}_n$ freely. Thus, by replacing x_i with x_1x_i if necessary, we may further suppose that $\chi_1(x_i) \neq 1$ for each i.

It is convenient to introduce a little more notation. Let 2^n denote the set of all functions on $\{1, 2, \ldots, n\}$ taking values 0 or 1. For each t in 2^n, define $e(t)$ in $L^1(\mathbf{F}_n)$ by $e(t) = (c_1\bar{1} - \bar{x}_1) * (c_2\bar{1} - \bar{x}_2) * \cdots * (c_n\bar{1} - \bar{x}_n)$, where $c_i = 1$ if $t(i) = 0$ and $c_i = \chi_1(x_i)$ if $t(i) = 1$. The constant functions in 2^n with values 0 and 1 will be denoted by 0 and 1 respectively. We will require the following lemma.

LEMMA. *Let t be in $2^n \backslash \{0, 1\}$. Then $e(t)$ is in I^2.*

PROOF. Since t is not constant, there is an i between 1 and $n - 1$ such that either

$$e(t) = \cdots * (\bar{1} - \bar{x}_i) * (\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}) * \cdots$$

or

$$e(t) = \cdots * \left(\chi_1(x_i)\bar{1} - \bar{x}_i\right) * \left(\bar{1} - \bar{x}_{i+1}\right) * \cdots .$$

Thus, since I^2 is an ideal in $L^1(\mathbf{F}_n)$, it will suffice to show that

$$\left(\bar{1} - \bar{x}_i\right) * \left(\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}\right) \quad \text{and} \quad \left(\chi_1(x_i)\bar{1} - \bar{x}_i\right) * \left(\bar{1} - \bar{x}_{i+1}\right)$$

are in I^2 for each i between 1 and $n - 1$.

For this, choose h in $L^1(\mathbf{F}_n)$ such that h is in $\ker \phi_1$ and $\bar{1} - h$ is in $\ker \phi_0$. Then

$$\begin{aligned}(1)\quad \left(\bar{1} - \bar{x}_i\right) * \left(\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}\right) &= h * \left(\bar{1} - \bar{x}_i\right) * \left(\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}\right) \\ &+ (\bar{1} - h) * \left(\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}\right) * \left(\bar{1} - \bar{x}_i\right) + (\bar{1} - h) * \left(\overline{x_i x_{i+1}} - \overline{x_{i+1} x_i}\right).\end{aligned}$$

Now $\bar{1} - \bar{x}_i$ has a square root defined by the binomial expansion

$$\left(\bar{1} - \bar{x}_i\right)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (\bar{x}_i)^{*k},$$

where the series converges because $\binom{1/2}{k}$ is an l^1-sequence and because $\|(\bar{x}_i)^{*k}\| = 1$ for each k. The square root is in $\ker \phi_0$ because $0 = \phi_0(\bar{1} - \bar{x}_i) = \phi_0((\bar{1} - \bar{x}_i)^{1/2})^2$. Hence

$$\begin{aligned} h * \left(\bar{1} - \bar{x}_i\right) * \left(\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}\right) &\\ = \left[h * \left(\bar{1} - \bar{x}_i\right)^{1/2}\right] &* \left[\left(\bar{1} - \bar{x}_i\right)^{1/2} * \left(\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}\right)\right] \\ \in [\ker \phi_1 * \ker \phi_0] &* [\ker \phi_0 * \ker \phi_1] \subseteq I^2. \end{aligned}$$

It may be shown in the same way that $(\bar{1} - h) * (\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1}) * (\bar{1} - \bar{x}_i)$ is in I^2. Finally,

$$\overline{x_i x_{i+1}} - \overline{x_{i+1} x_i} = \left(\bar{1} - \left(x_{i+1} x_i x_{i+1}^{-1} x_i^{-1}\right)^{-}\right) * \left(\overline{x_i x_{i+1}}\right),$$

where $\bar{1} - (x_{i+1} x_i x_{i+1}^{-1} x_i^{-1})^{-}$ is in I and has a square root which must also be in I because $I = \ker \phi_0 \cap \ker \phi_1$. Hence $\overline{x_i x_{i+1}} - \overline{x_{i+1} x_i}$ is in I^2 also. It now follows from (1) that $(\bar{1} - \bar{x}_i) * (\chi_1(x_{i+1})\bar{1} - \bar{x}_{i+1})$ is in I^2 for each i. That $(\chi_1(x_i)\bar{1} - \bar{x}_i) * (\bar{1} - \bar{x}_{i+1})$ is in I^2 also may be proved in the same way.

We now continue the proof of the theorem. Every x in $\mathbf{F}_n$ may be written uniquely in the form $x = x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} y$, where $k_1, k_2, \ldots, k_n$ are integers and y is in $\mathbf{F}_n'$, the commutator subgroup of $\mathbf{F}_n$. For each i between 1 and n, let $Z_i = \{x_i^{k_i} \cdots x_n^{k_n} y \mid k_i, \ldots, k_n \in \mathbf{Z},\ y \in \mathbf{F}_n'\}$ and let $\langle x_i \rangle$ be the subgroup of $\mathbf{F}_n$ generated by x_i. Let $Z_{n+1} = \mathbf{F}_n'$. Then $Z_1 = \mathbf{F}_n$ and $Z_{i+1} \subseteq Z_i$, $i = 1, 2, \ldots, n$.

Let i be between 1 and n and denote by $L^1(\langle x_i \rangle)$ and $L^1(Z_i)$ the spaces of functions in $L^1(\mathbf{F}_n)$ with support in $\langle x_i \rangle$ and Z_i respectively. Since $\chi_1(x_i) \neq 1$, $L^1(\langle x_i \rangle) \cap I$ has codimension two in $L^1(\langle x_i \rangle)$. Hence we may define bounded linear functionals η, ξ on $L^1(\langle x_i \rangle)$ and a bounded linear operator $T\colon L^1(\langle x_i \rangle) \to L^1(\langle x_i \rangle) \cap I$ in a unique way such that for each f in $L^1(\langle x_i \rangle)$,

$$(2) \qquad f = \eta(f)\left(\bar{1} - \bar{x}_i\right) + \xi(f)\left(\chi_1(x_i)\bar{1} - \bar{x}_i\right) + T(f).$$

Furthermore, $L^1(\langle x_i\rangle)$ is a subalgebra of $L^1(\mathbf{F}_n)$ isomorphic to $L^1(\mathbf{Z})$ (the group algebra of the integers) and $L^1(\langle x_i\rangle) \cap I$ is a closed ideal in $L^1(\langle x_i\rangle)$. Hence, by [**5**, Theorem 2], $L^1(\langle x_i\rangle) \cap I$ has a bounded approximate identity. Let J_i be the closed linear span of $(L^1(\langle x_i\rangle) \cap I) * L^1(\mathbf{F}_n)$. Then it is clear that J_i is a left Banach module over $L^1(\langle x_i\rangle) \cap I$, that J_i is contained in I, and that $L^1(\langle x_i\rangle) \cap I$ has a bounded approximate identity for J_i (in the sense of [**1**, 11.8]).

Now for each f in $L^1(Z_i)$ and each z in Z_{i+1}, let f_z be the function in $L^1(\langle x_i\rangle)$ defined by $f_z(x_i^k) = f(x_i^k z)$. Then $\Sigma_{z\in Z_{i+1}}\|f_z\| = \|f\|$ and $f = \Sigma_{z\in Z_{i+1}} f_z * \bar{z}$. Hence, by (2),

$$f = (\bar{1} - \bar{x}_i) * \Big(\sum_{z\in Z_{i+1}} \eta(f_z)\bar{z}\Big) + (\chi_1(x_i)\bar{1} - \bar{x}_i) * \Big(\sum_{z\in Z_{i+1}} \xi(f_z)\bar{z}\Big) + \sum_{z\in Z_{i+1}} T(f_z) * \bar{z}.$$

Put $h = \Sigma_{z\in Z_{i+1}} T(f_z) * \bar{z}$. Then h is in J_i and so, by Cohen's factorization theorem [**1**, 11.10], there are a in $L^1(\langle x_i\rangle) \cap I$ and h' in J_i such that $h = a * h'$. Since both $L^1(\langle x_i\rangle) \cap I$ and J_i are contained in I, it follows that h is in I^2. Hence, putting $\Sigma_{z\in Z_{i+1}}\eta(f_z)\bar{z} = g_0$ and $\Sigma_{z\in Z_{i+1}}\xi(f_z)\bar{z} = g_1$, we have shown that for each f in $L^1(Z_i)$,

$$f = (\bar{1} - \bar{x}_i) * g_0 + (\chi_1(x_i)\bar{1} - \bar{x}_i) * g_1 + h, \tag{3}$$

where g_0 and g_1 are in $L^1(Z_{i+1})$ and h is in I^2.

In particular, if f is in $L^1(\mathbf{F}_n)$, then

$$f = (\bar{1} - \bar{x}_1) * g_0 + (\chi_1(x_1)\bar{1} - \bar{x}_1) * g_1 + h,$$

where g_0 and g_1 are in $L^1(Z_2)$ and h is in I^2. Next, applying (3) to g_0 and g_1 and remembering that I^2 is an ideal, we find that

$$\begin{aligned} f = {} & (\bar{1} - \bar{x}_1) * (\bar{1} - \bar{x}_2) * g_{00} + (\bar{1} - \bar{x}_1) * (\chi_1(x_2)\bar{1} - \bar{x}_2) * g_{01} \\ & + (\chi_1(x_1)\bar{1} - \bar{x}_1) * (\bar{1} - \bar{x}_2) * g_{10} \\ & + (\chi_1(x_1)\bar{1} - \bar{x}_1) * (\chi_1(x_2)\bar{1} - \bar{x}_2) * g_{11} + h'', \end{aligned}$$

where $g_{00}, g_{01}, g_{10}, g_{11}$ are in $L^1(Z_3)$ and h'' is in I^2. Now applying (3) to g_{00}, g_{01}, g_{10} and g_{11} and so on, we find after n steps that for each f in $L^1(\mathbf{F}_n)$,

$$f = \sum_{t\in 2^n} e(t) * g(t) + \psi,$$

where $g(t)$ is in $L^1(Z_{n+1})$ $(= L^1(\mathbf{F}_n'))$ for every t in 2^n and ψ is in I^2. Finally, by the lemma, $e(t)$ is in I^2 for each t other than 0 or 1 and so for every f in $L^1(\mathbf{F}_n)$,

$$f = e(0) * g(0) + e(1) * g(1) + \psi',$$

where $g(0)$ and $g(1)$ are in $L^1(\mathbf{F}_n')$ and ψ' is in I^2.

Let f be in I. Then,

$$0 = \phi_1(f) = \phi_1(e(0))\phi_1(g(0)) + \phi_1(e(1))\phi_1(g(1)) + \phi_1(\psi').$$

Since $\phi_1(\psi') = 0 = \phi_1(e(1))$ and $\phi_1(e(0)) \neq 0$, it follows that $\phi_1(g(0)) = 0$. Thus,

$$0 = \sum_{x\in\mathbf{F}_n} g(0)(x)\chi_1(x) = \sum_{y\in\mathbf{F}_n'} g(0)(y),$$

because $g(0)$ is supported in $\mathbf{F}_n'$ and characters are trivial on $\mathbf{F}_n'$. Hence $g(0)$ is in $I_0(\mathbf{F}_n')$. Similarly, using the fact that $\phi_0(f) = 0$, it may be shown that $g(1)$ is in $I_0(\mathbf{F}_n')$. Now $I_0(\mathbf{F}_n')$ is contained in I and is idempotent by [6]. Hence,

$$g(0), g(1) \in I_0(\mathbf{F}_n') = I_0(\mathbf{F}_n')^2 \subseteq I^2,$$

and so f is in I^2. Therefore I is idempotent.

There are still many unanswered questions concerning factorization in finite-codimensional ideals of group algebras. Some of these questions are discussed in more detail in §4 of [7]. It is mentioned there, without proof, that the above theorem may be used to show that, if G is finitely generated, then every ideal I such that $L^1(G)/I$ is commutative and finite dimensional is idempotent. In particular it follows that, if G is finitely generated, then every ideal with codimension three in $L^1(G)$ is idempotent. We now give a proof of this fact.

THEOREM 2. *Let A be a complex algebra such that $A^2 = A$. Suppose that*
(i) *every ideal with codimension one in A is idempotent; and*
(ii) *if I and J are ideals with codimension one in A, then $IJ = JI$.*
Then every ideal I in A such that A/I is finite dimensional and commutative is idempotent.

PROOF. By adjoining an identity we may assume that A has a unit, which we will denote by 1. Let I be an ideal in A with A/I commutative and finite dimensional, but with $\dim(A/I) \geqslant 2$. We may suppose that every ideal properly containing I is idempotent.

Denote by ρ the quotient map from A to A/I and let R be the radical of A/I. If $R \neq (0)$, then $R^2 \subsetneq R$ because R is a finite-dimensional radical algebra. Hence, if $R \neq (0)$, $(\rho^{-1}(R))^2 \subseteq \rho^{-1}(R^2) \neq \rho^{-1}(R)$, and so $\rho^{-1}(R)$ is a nonidempotent ideal in A which properly contains I. Therefore $R = (0)$, and so A/I is semisimple. It follows that A/I is isomorphic to the direct sum of n copies of $\mathbf{C}$, where n is the codimension of I. Hence, there are ideals $I_1, I_2, \ldots, I_n$ with codimension one in A such that $I = \bigcap_{j=1}^n I_j$.

Since I_1 and I_2 are distinct codimension one ideals in A, there is an h in I_1 such that $1 - h$ is in I_2. Hence, for each f in I, $f = fh + f(1-h)$, and so $I \subseteq II_1 + II_2$.

Similarly, if $j_1, j_2, \ldots, j_k$ are distinct integers between 1 and n, where $k \leqslant n-2$, and i_1 and i_2 are distinct integers not equal to any of the j_k's then

$$II_{j_1}I_{j_2}\cdots I_{j_k} \subseteq II_{j_1}\cdots I_{j_k}I_{i_1} + II_{j_1}\cdots I_{j_k}I_{i_2},$$

where now $\{j_1, j_2, \ldots, j_k, i_1\}$ and $\{j_1, j_2, \ldots, j_k, i_2\}$ are sets of distinct integers. We may thus show that I is contained in the sum of ideals of the form $II_{j_2}I_{j_3}\cdots I_{j_n}$, where $j_2, j_3, \ldots, j_n$ are distinct integers between 1 and n. However, $I \subseteq \bigcap_{j=1}^n I_j$, and so $II_{j_2}I_{j_3}\cdots I_{j_n} \subseteq I_{j_1}I_{j_2}\cdots I_{j_n}$, where j_1 is chosen to be the integer between 1 and n which does not appear in the list $j_2, j_3, \ldots, j_n$. Therefore,

$$I \subseteq \sum_{\pi \in S_n} \left(I_{\pi(1)}I_{\pi(2)}\cdots I_{\pi(n)}\right), \tag{4}$$

where S_n is the group of permutations of $\{1, 2, \ldots, n\}$.

Now,

$$\begin{aligned} I_1 I_2 \cdots I_n &= (I_1)^2 (I_2)^2 \cdots (I_n)^2, \quad \text{by (i)}, \\ &= I_1 I_2 I_1 I_2 (I_3)^2 \cdots (I_n)^2, \quad \text{by (ii)}, \\ &= (I_1 I_2 I_3)^2 (I_4)^2 \cdots (I_n)^2, \quad \text{by (ii) applied twice}, \\ &= \cdots \\ &= (I_1 I_2 \cdots I_n)^2 \subseteq I^2. \end{aligned}$$

Similarly, $I_{\pi(1)} I_{\pi(2)} \cdots I_{\pi(n)} \subseteq I^2$ for every π in S_n. Therefore, by (4), $I \subseteq I^2$.

REMARKS. (a) The codimension one ideals of A will satisfy (ii) either if A is commutative (clear) or if the codimension two ideals of A are idempotent ($IJ \subseteq I \cap J = (I \cap J)^2 \subseteq JI$).

(b) If A is a Banach algebra and $A^2 = A$, then every codimension one ideal in A is closed. Hence we need only verify that (i) and (ii) hold for the *closed* codimension one ideals of A. Furthermore, in the course of the proof it was shown that if A satisfies (i) and (ii) and if I is an ideal in A such that A/I is commutative and finite dimensional, then I is the intersection of codimension one ideals. Hence under these conditions I is closed.

(c) Suppose that A satisfies the conditions of the theorem, then all ideals with codimension two or three in A are idempotent. If I has codimension two, then A/I is semisimple (otherwise the inverse image of $\mathrm{Rad}(A/I)$ will be a nonidempotent codimension one ideal). Hence A/I is isomorphic to $\mathbf{C} \oplus \mathbf{C}$ (the only semisimple, two-dimensional complex algebra). Now $\mathbf{C} \oplus \mathbf{C}$ is commutative and so I is idempotent by the theorem. A similar argument proves the result for codimension three ideals. The result does not necessarily hold for codimension four ideals because there is a four-dimensional noncommutative, semisimple complex algebra—namely the 2×2 complex matrices.

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