A NOTE ON NEW GENERATING RELATIONS FOR FUNCTIONS OF SEVERAL VARIABLES

D. P. SHUKLA¹

Abstract. In this paper we have derived generating relations for functions of several variables by using the operator \( T_k = x(k + xD) \) and the operational relations involving this operator. Some recent results of Srivastava and Panda [7] have been conveniently obtained by this method as well as some hitherto unknown results established.

Introduction. Lauricella's function of several variables is defined as follows

\[
F_B^{(r)}[(\alpha; m_i); (\beta_i; \phi_i); (\gamma; \Psi_i); x_1, \ldots, x_r]
= \sum_{k_1, \ldots, k_r = 0}^{\infty} \frac{(\alpha)_{\epsilon m, k_i} (\beta_1)_{k_1} \phi_1 \cdots (\beta_r)_{k_r} \phi_r \gamma^{k_1} \cdots k_r^{k_r}}{k_1! \cdots k_r!},
\]

where the parameters \( \alpha, \beta, \gamma \) are arbitrary complex numbers, the coefficients \( m_i, \phi_i, \Psi_i \) are real and positive and for the convergence of multiple series

\[
1 + \Psi_i - m_i - \phi_i \geq 0, \quad i = 1, 2, \ldots, r,
\]

the equality holds only if \( |x_1|, |x_2|, \ldots, |x_r| \) are constrained appropriately. In particular if \( m_i = m, \Psi_i = \Psi, i = 1, 2, \ldots, r \), then the equality in (1.2) will hold when either

\[
m > \Psi \quad \text{and} \quad \sum_{i=1}^{r} \left( \frac{|x_i|}{G_i} \right)^{1/(m-\Psi)} < 1
\]

or

\[
m \leq \Psi \quad \text{and} \quad \max_{1 \leq i \leq r} \left( \frac{|x_i|}{G_i} \right) < 1,
\]

where

\[
G_i = m^{-m\phi_i} \Psi_i, \quad i = 1, 2, \ldots, r.
\]

¹ Research supported by Council of Scientific and Industrial Research Government of India, Grant No. 13 (2707-A)/Pool.

©1983 American Mathematical Society

0002-9939/83/0000-1479/$02.75
In this paper we have found new generating relations for Lauricella’s function and some related function of several variables, viz.

\[
\sum_{n_1, \ldots, n_r = 0}^{\infty} \frac{(\lambda_1)_{2n_1} \cdots (\lambda_r)_{2n_r} t_1^{n_1} \cdots t_r^{n_r}}{(\lambda_1)_n \cdots (\lambda_r)_n n_1! \cdots n_r!} \cdot F_D^{(r)}[(\lambda; \; m_i); (-n; \; m_i); (\gamma; \; \Psi_i); x_1, \ldots, x_r]
\]

\[
= \prod_{i=1}^{r} \left\{ \left[ 1 + (1 - 4t_i)^{-1/2} \right]^{\lambda_i - 1} \left[ \frac{2}{1 + (1 - 4t_i)^{1/2}} \right] \right\} \cdot F_D^{(r)}[(\lambda; \; m_i); (-n_i - 2n_i; \; m_i); (\gamma; \; \Psi_i); x_1 \left( \frac{2t_1}{1 + (1 - 4t_1)^{1/2}} \right)^{m_1}, \ldots, x_r \left( \frac{2t_r}{1 + (1 - 4t_r)^{1/2}} \right)^{m_r}].
\]

Recently Srivastava and Panda [7] considered the functions

\[
f[z_1, \ldots, z_r] = \sum_{k_1, \ldots, k_r = 0}^{\infty} C(k_1, \ldots, k_r) \frac{z_1^{k_1}}{k_1!} \cdots \frac{z_r^{k_r}}{k_r!},
\]

where the coefficients \( C(k_1, \ldots, k_r), k_i \geq 0, i = 1, 2, \ldots, r, \) are arbitrary constants real or complex.

\[
\Delta_{n_1, \ldots, n_r; q_1, \ldots, q_r}(z_1, \ldots, z_r) = \sum_{k_1 = 0}^{[n_1/q_1]} \cdots \sum_{k_r = 0}^{[n_r/q_r]} C(k_1, \ldots, k_r)
\]

\[
\cdot \prod_{i=1}^{r} \left\{ \frac{(-n_i)_{q_i} z_i^{k_i}}{(1 + \alpha_i + \beta_i n_i)_{q_i k_i} k_i!} \right\}
\]

where \( \alpha_i \) and \( \beta_i \) are parameters independent of \( n_1, \ldots, n_r \) and \( q_i \), are arbitrary positive integers, \( i = 1, \ldots, r \) and

\[
S_{n_1, \ldots, n_r; q_1, \ldots, q_r}(\lambda_1, \ldots, \lambda_r; z_1, \ldots, z_r) = \sum_{k_1 = 0}^{[n_1/q_1]} \cdots \sum_{k_r = 0}^{[n_r/q_r]} C(k_1, \ldots, k_r)
\]

\[
\cdot \prod_{i=1}^{r} \left\{ \frac{(-n_i)_{q_i k_i}(1 + \alpha_i + (\beta_i + 1)n_i)_{\lambda_i k_i} z_i^{k_i}}{(1 + \alpha_i + \beta_i n_i)_{(\lambda_i + q_i) k_i} k_i!} \right\}
\]

where \( \alpha_i, \beta_i \) and \( \lambda_i, i = 1, 2, \ldots, r, \) are complex parameters independent of \( n_1, \ldots, n_r \) and \( q_1, \ldots, q_r \), are arbitrary positive integers and derived the generating relation for them.

Besides these Srivastava [5, 6], Sharma and Abiodun [1] and Carlitz and Srivastava [10] derived generating relations for (1.7) and the \( G \) function and obtained generating functions for Lauricella’s function of several variables. Shukla [2] using the operator \( T_k = x(k + xD) \) and the operational relations involving these operators derived the results of Srivastava [6], Sharma and Abiodun [1] and Carlitz and Srivastava [10] and established some hitherto unknown results.
In this paper we propose to derive the results of Srivastava and Panda [7] and other results by making use of operator formulas (2.1), (3.1), (3.4), (3.7) and (3.9).

In §2 we obtain generating relations for functions defined by (1.8) and (1.9) and in §3 we derive various generating functions for Lauricella’s function of several variables.

2. Generating relations for functions of several variables. In this section we shall use the Mittal [4] operational generating formula

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T_{a+1}^{n} (f(x)) = \frac{(1 + \nu)^{a+1}}{1 - (m-1)\nu} f[x(1 + \nu)],
\]

where \( \nu = x(1 + \nu)^m \), \( m \) being constant and \( f(x) \) admits a formal power series in \( x \) and \( T_k \equiv x(k + xD), D = d/dx \).

Assuming that the operator \( T_{1,k} \) is the operator \( T_k \) operating on \( t_1 \) alone and similarly \( T_{r,k} \) the operator on \( t_r \) alone and replacing \( \alpha \) by \( \alpha_1 = \alpha_1, \ldots, \alpha_r = \alpha_r, m \) by \( m_1 = \beta_1 + 1, \ldots, m_r = \beta_r + 1, n \) by \( n_1, n_2, \ldots, n_r \) respectively and putting

\[
f(x) \equiv f[x_1(-y_1)^{-\beta_1}t_1^{\beta_1+1}]^{m_1}, \ldots, x_r(-y_r)^{-\beta_r}t_r^{\beta_r+1}]^{m_r}
\]

in (2.1) \( r \) times, we easily get

\[
\sum_{n_1, \ldots, n_r = 0}^{\infty} \left( \alpha_1 + (\beta_1 + 1)n_1 \right) \cdots \left( \alpha_r + (\beta_r + 1)n_r \right) t_1^{n_1} t_2^{n_2} \cdots t_r^{n_r} \cdot \sum_{k_1, \ldots, k_r = 0}^{N} C(k_1, \ldots, k_r) \prod_{i=1}^{r} \frac{(-n_i)_{m_i,k_i}}{(1 + \alpha_i + \beta_i n_i)_{m_i,k_i}} \frac{x_i^{k_i} y_i^{-\beta_i} t_i^{\beta_i+1}}{k_i!} = \frac{(1 + \nu_1)^{\alpha_1+1} \cdots (1 + \nu_r)^{\alpha_r+1}}{(1 - \beta_1 \nu_1) \cdots (1 - \beta_r \nu_r)} f[x_1(-y_1)^{-\beta_1}t_1^{\beta_1+1}(1 + \nu_1)^{\beta_1+1}]^{m_1}, \ldots, x_r(-y_r)^{-\beta_r}t_r^{\beta_r+1}(1 + \nu_r)^{\beta_r+1}]^{m_r}],
\]

where \( \nu_i = t_i(1 + \nu_i)^{\beta_i+1}, \) \( i = 1, 2, \ldots, r, \) and \( N = \min(n_1, n_2, \ldots, n_r). \) Now putting \( y_i = t_i, \) \( i = 1, 2, \ldots, r, \) in (2.2) we get

\[
\sum_{n_1, \ldots, n_r = 0}^{\infty} \left( \alpha_1 + (\beta_1 + 1)n_1 \right) \cdots \left( \alpha_r + (\beta_r + 1)n_r \right) \Delta^{(a_1, \ldots, a_r; \beta_1, \ldots, \beta_r)}(x_1, \ldots, x_r) t_1^{n_1} \cdots t_r^{n_r} = \frac{(1 + \nu_1)^{\alpha_1+1} \cdots (1 + \nu_r)^{\alpha_r+1}}{(1 - \beta_1 \nu_1) \cdots (1 - \beta_r \nu_r)} f[x_1(-\nu_1)^{m_1}, \ldots, x_r(-\nu_r)^{m_r}]
\]

where the function \( \Delta \) is given by (1.8), which is due to Srivastava and Panda [7, Theorem 2]. Further with the similar assumptions on operator \( T_k, \alpha \) and \( m \) as in the derivation of (2.3), but taking

\[
f(x) \equiv f[x_1(-1)^{m_1} y_1^{\beta_1} t_1^{\beta_1+1} (1 + \beta_1 m_1 + \lambda_1), \ldots, x_r(-1)^{m_r} y_r^{-\beta_r} t_r^{\beta_r+1} (1 + \beta_r m_r + \lambda)]
\]
in (2.1) and using it \( r \) times, we similarly have

\[
\sum_{n_1, \ldots, n_r=0}^{\infty} \left( \alpha_1 + (\beta_1 + 1)n_1 \right) \cdots \left( \alpha_r + (\beta_r + 1)n_r \right) S^{(a_1, \ldots, a_r; \beta_1, \ldots, \beta_r)}_{n_1, \ldots, n_r} [x_1, \ldots, x_r] t_1^{n_1} \cdots t_r^{n_r} \\
= \prod_{i=1}^{r} \left[ \frac{(1 + \nu_i)^{a_i+1}}{(1 - \beta_i \nu_i)} \right] f \left[ x_i (-\nu_i)^{m_1} (1 + \nu_i)^{\lambda_i}, \ldots, x_r (-\nu_r)^{m_r} (1 + \nu_r)^{\lambda_r} \right],
\]

where \( \nu_i = \nu_i (1 + \nu_i)^{\beta_i+1} \), \( i = 1, 2, \ldots, r \), and the function \( S \) is given by (1.9). This is also due to Srivastava and Panda [7, Theorem 2.1].

3. Lauricella's function of several variables. We now make use of the Mittal [3] operational formula

\[
\sum_{n=0}^{\infty} \frac{1}{n!} T^n_{a+1} \{ f(x) \} = (1 - x)^{-a-1} f \left[ \frac{x}{1-x} \right],
\]

where \( f(x) \) admits a formal power series in \( x \). Replacing \( a \) by \( a_1 = \lambda_1 - 1, \ldots, a_r = \lambda_r - 1, n \) by \( n_1, \ldots, n_r \) and assuming that \( T_{i,k} \) is the operator \( T_k \) operating on \( t_i \) alone for \( i = 1, 2, \ldots, r \) respectively and putting

\[
f(x) = F^{(r)}_D \left[ (\lambda; m_i); (\beta_i; \phi_i); (\gamma; \Psi_i); (-t_i)^{m_i}; x_1, \ldots, x_r \right]
\]

in (3.1) and using it \( r \) times, we get

\[
\prod_{i=1}^{r} \left[ \frac{(1 - t_i)^{\lambda_i}}{(1 - \beta_i \phi_i)} \right] F^{(r)}_D \left[ (\lambda; m_i); (\beta_i; \phi_i); (\gamma; \Psi_i); x_1 \left( \frac{t_1}{t_1 - 1} \right)^{m_i}, \ldots, x_r \left( \frac{t_r}{t_r - 1} \right)^{m_r} \right].
\]

Taking \( \beta_r = \lambda_r \) and \( \phi_r = m_r \) in (3.2) we have the generating relation

\[
\prod_{i=1}^{r} \left[ \frac{(1 - t_i)^{\lambda_i}}{(1 - \beta_i \phi_i)} \right] F^{(r)}_D \left[ (\lambda; m_i); (\gamma; \Psi_i); x_1 \left( \frac{t_1}{t_1 - 1} \right)^{m_i}, \ldots, x_r \left( \frac{t_r}{t_r - 1} \right)^{m_r} \right].
\]
Again by making use of the Mittal [3] operational formula

\[ \sum_{n=0}^{\infty} \frac{1}{n!} T_{a+n}^{n}(f(x)) = (1 - 4x)^{-1/2} \left[ \frac{2}{1 + (1 - 4x)^{1/2}} \right]^{a-1} \]

replacing \( a \) by \( a_1 = \lambda_1, \ldots, a_r = \lambda_r, n \) by \( n_1, \ldots, n_r \) and assuming that \( T_{i,k} \) is the operator \( T_{i,k} \) operating on \( t_i \) alone, \( i = 1, 2, \ldots, r \), respectively and taking

\[ f(x) = F_D^{(r)}[(\lambda; m_i); (\beta_i; \phi_i); (\gamma; \Psi_i); x_1 t_1^{m_1}, \ldots, x_r t_r^{m_r}] \]

in (3.4) and using it \( r \) times, we have the generating relation

\[ \sum_{n_1, \ldots, n_r=0}^{N} \frac{(\lambda_1)_{2n_1} \cdots (\lambda_r)_{2n_r} t_1^{n_1} \cdots t_r^{n_r}}{(\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} n_1! \cdots n_r!} 
\]

\[ \prod_{i=1}^{r} \left[ (1 - 4t_i)^{-1/2} \left[ \frac{2}{1 + (1 - 4t_i)^{1/2}} \right]^{\lambda_i - 1} \right] \cdot F_D^{(r)}[(\lambda; m_i); (\beta_i; \phi_i); (\gamma; \Psi_i); x_1 \left( \frac{2t_1}{1 + (1 - 4t_1)^{1/2}} \right)^{m_1}, \ldots, \]

\[ x_r \left( \frac{2t_r}{1 + (1 - 4t_r)^{1/2}} \right)^{m_r}] \]

Putting \( \phi_r = m_r \) and \( \beta_r = 1 - \lambda_r - 2n_r \) in equation (3.5), we get

\[ \sum_{n_1, \ldots, n_r=0}^{\infty} \frac{(\lambda_1)_{2n_1} \cdots (\lambda_r)_{2n_r} t_1^{n_1} \cdots t_r^{n_r}}{(\lambda_1)_{n_1} \cdots (\lambda_r)_{n_r} n_1! \cdots n_r!} F_D^{(r)}[(\lambda; m_i); (-n_i; m_i); (\gamma; \Psi_i); x_1, \ldots, x_r] \]

\[ = \prod_{i=1}^{r} \left[ (1 - 4t_i)^{-1/2} \left[ \frac{2}{1 + (1 - 4t_i)^{1/2}} \right]^{\lambda_i - 1} \right] \cdot F_D^{(r)}[(\lambda; m_i); (1 - \lambda_i - 2n_i; m_i); (\gamma; \Psi_i); \]

\[ x_1 \left( \frac{2t_1}{1 + (1 - 4t_1)^{1/2}} \right)^{m_1}, \ldots, x_r \left( \frac{2t_r}{1 + (1 - 4t_r)^{1/2}} \right)^{m_r}] \]
Again, making use of the Mittal [3] operational formula

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_n \{ f(x) \} = (1 + x)^{a-1} f[x(1 + x)], \]

and proceeding as above, we get the generating relation

\[ \sum_{n_1, \ldots, n_r=0}^{\infty} (-1)^{n_1+\cdots+n_r} \prod_{i=1}^{r} \frac{(1 - \lambda_i)_{n_i} t_i^{n_i}}{n_i!} \]

\[ \cdot \binom{\lambda}{m_i} : (-n; m_i), (\lambda_i; m_i); x_1, \ldots, x_r \]

\[ (\gamma; \Psi_i) \]

\[ = \prod_{i=1}^{r} (1 + t_i)^{\lambda_i-1} \frac{1}{F_D^{(r)}} \left[ (\lambda; m_i) : (\lambda_i - n_i; 2m_i); (\gamma; \Psi_i); \right. \]

\[ \left. x_1(-t_1(1 + t_1)^{m_1}), \ldots, x_r(-t_r(1 + t_r)^{m_r}) \right]. \]

Next, using the Mittal [3] operational formula

\[ \sum_{n=0}^{\infty} \frac{1}{n!} \Gamma_n \{ f(x) \} = (1 + 4x)^{-1/2} \frac{2}{1 + (1 + 4x)^{1/2}} \]

\[ f \left[ \frac{x + (1 + 4x)^{1/2}}{2} \right], \]

replacing \( a \) by \( a_1 = \lambda_1, \ldots, a_r = \lambda_r \), \( n \) by \( n_1, \ldots, n_r \) and assuming that \( T_{i,k} \) is the operator \( T_k \) operating on \( t_i \) alone, \( i = 1, 2, \ldots, r \), respectively and putting

\[ f(x) \equiv F_D^{(r)} \left[ (\lambda; m_i) : (\beta_i; \phi_i); (\gamma; \Psi_i); x_1 \left( \frac{27}{4} t_1 \right)^{m_1}, \ldots, \left( \frac{27}{4} t_r \right)^{m_r} \right] \]

in (3.9) and using (3.9) \( r \)-times we get the following generating relation:

\[ \sum_{n_1, \ldots, n_r=0}^{\infty} \prod_{i=1}^{r} \frac{(1 - \lambda_i)_{2n_i} t_i^{n_i}}{n_i! (1 - \lambda_i)_{n_i}} \]

\[ \cdot \sum_{k_1, \ldots, k_r=0}^{n_1} \frac{(\lambda)_{em, k_i}(\beta_1)_{k_1, \phi_1} \cdots (\beta_r)_{k_r, \phi_r}}{\gamma_{em, k_i}} \frac{x_1^{k_1} \cdots x_r^{k_r}}{k_1! \cdots k_r!} \]

\[ \cdot \prod_{i=1}^{r} \left[ \frac{(\lambda - n_i)/2}{m_i, k_i} \frac{(\lambda - n_i + 1)/2}{m_i, k_i}; \frac{(\lambda - 2n_i + 3)/3}{m_i, k_i} \right] \]

\[ = \prod_{i=1}^{r} \left( 1 + 4t_i \right)^{-1/2} \frac{2}{1 + (1 + 4t_i)^{1/2}} \left[ \left( \frac{27}{4} t_i \right)^{1/2} \right]^{\lambda_i} \]

\[ \cdot F_D^{(r)} \left[ (\lambda; m_i) : (\beta_i; \phi_i); (\gamma; \Psi_i); x_1 \left( \frac{27}{4} t_1 \right)^{1/2} \right], \ldots, \]

\[ x_r \left( \frac{27}{4} t_r \right)^{1/2} \right]. \]
which is a generalization of a result due to Shukla [2]. Finally replacing \( a \) by \( a_i = \lambda_i - 1, \ldots, a_r = \lambda_r - 1, n \) by \( n_1, \ldots, n_r \) and assuming that \( T_{i,k} \) is the operator \( T_k \) operating on \( t_i \) alone, \( i = 1, 2, \ldots, r \) respectively and putting
\[
f(x) \equiv F^{(r)}_D \left[ (\lambda; m_i): (\beta_i; \phi_i); (\gamma; \Psi_i); x_1 \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_1 \right]^{m_1}, \ldots, x_r \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_r \right]^{m_r} \right]
\]
in (2.1) and using it \( r \)-times we get the generating relation
\[
(3.11) \quad \sum_{n_1, \ldots, n_r = 0}^{\infty} (-1)^{n_1 + \cdots + n_r} \prod_{i=1}^{r} \frac{(\lambda_i)^{m_i n_i} t_i^{n_i}}{n_i! (\lambda_i)^{(m-1)n_i}} \sum_{k_1, \ldots, k_r = 0}^{\infty} \frac{(\lambda e_{m,k})(\beta_1)_{k_1,\phi_1} \cdots (\beta_r)_{k_r,\phi_r} x_1^{k_1} \cdots x_r^{k_r}}{(\gamma)_{k_1,\phi_1,\psi_1} k_1! \cdots k_r!} \prod_{i=1}^{r} \left[ \frac{(-n_i)_{m_i, k_i, n_i + mn_i}}{(\lambda + mn_i)_{m_i, k_i}} \cdots \frac{((\lambda + (m-1)n_i) / (2 - m))_{m_i, k_i}}{((\lambda + (m-1)n_i + 1 - m) / (2 - m))_{m_i, k_i}} \right] = \prod_{i=1}^{r} \frac{(1 + \nu_i)^{\lambda_i}}{(1 - (m-1)\nu_i)^{\lambda_i}} F^{(r)}_D \left[ (\lambda; m_i): (\beta_i; \phi_i); (\gamma; \Psi_i); x_1 \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_1 (1 + \nu_i) \right]^{m_1}, \ldots, x_r \left[ \frac{(2 - m)^{2-m}}{(1 - m)^{1-m}} t_r (1 + \nu_i) \right]^{m_r} \right],
\]
where \( \nu_i = t_i (1 + \nu_i)^{m_i}, i = 1, 2, \ldots, r \), which is again a generalization of a result due to Shukla [2].

REFERENCES


Department of Mathematics, Indian Institute of Technology, Kampur-208016, India