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ON n-WIDTHS OF CERTAIN FUNCTIONAL CLASSES DEFINED BY LINEAR DIFFERENTIAL OPERATORS

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ABSTRACT. Let $A = D^r + \sum_{k=0}^{r-1} a_k(t)D^k$, $a_k \in C^k$, be a differential operator and let $W_p(A)$ be the class of functions $x(t)$ for which $\|Ax\| \leq 1$ in $L_p[0,1]$. We prove that the asymptotic behavior of the Kolmogorov widths $d_n(W_p(A), L_q)$, $1 \leq p, q \leq \infty$, when $n \to \infty$ does not depend on a_k.

Let L_p^r be the class of functions $x(t)$ having $x^{(r-1)}$ absolutely continuous on $[0,1]$ and $x^{(r)} \in L_p[0,1]$. Suppose

$$A = D^r + \sum_{k=0}^{r-1} a_k(t) D^k \tag{1}$$

is a linear differential operator, $a_k \in C^k[0,1]$ and $W_p(A) = \{x : x \in L_p^r, \|Ax\|_p \leq 1\}$. We shall write $W_p(A) = W_p^r$ if $A = D^r$.

The Kolmogorov n-width of a set W in a normed linear space X is defined as

$$d_n(W, X) = \inf_{\Gamma_n} \sup_{x \in W} \inf_{\gamma \in \Gamma_n} \|x - \gamma\|_X,$$

where the last infimum is taken over all n-dimensional linear subspaces $\Gamma_n \subset X$.

The Gelfand n-width of W is defined as

$$d^n(W, X) = \inf_{\Gamma^n} \sup_{x \in W \cap \Gamma^n} \|x\|_X$$

where the infimum is taken over all linear subspaces of codimension n.

The widths $d_n(W_p(A), L_q)$ were studied in several papers [**1–4**]. In particular, the precise rate of decrease of d_n when $n \to \infty$ was found in [**2**] for $p = q = \infty$ and in [**4**] for $p = q = 2$ and $p = 1$, $q = 2$.

It turned out, in these cases, that the asymptotic behavior of $d_n(W_p(A), L_q)$ is precisely the same for all operators A of the form (1). The purpose of this paper is to prove that this fact is true for all combinations of p and q.

THEOREM. *For all p, q, $1 \leq p, q \leq \infty$, and A*

$$\lim_{n \to \infty} \frac{d_n(W_p(A), L_q)}{d_n(W_p^r, L_q)} = 1. \tag{2}$$

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A remarkable feature of the simple proof given below is that it requires neither explicit construction of optimal subspaces Γ_n nor even any specific information on their structure.

Let L^r_{p0}, L^r_{p1} be subsets of L^r_p consisting of functions $x(t)$ for which $x^{(k)}(0) = 0$ or $x^{(k)}(1) = 0$ respectively $(k = 0, 1, \ldots, r-1)$; $\tilde{L}^r_p = L^r_{p0} \cap L^r_{p1}$; $\overset{0}{W}_p(A) = W_p(A) \cap L^r_{p0}$, $\overset{1}{W}_p(A) = W_p(A) \cap L^r_{p1}$, $\tilde{W}_p(A) = W_p(A) \cap \tilde{L}^r_p$. We will write $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$; $C_1, C_2, \ldots$ will denote positive constants which may depend on A, r, p, q but do not depend on n.

We shall need the following lemmas.

LEMMA 1. *There exists such C_1, C_2 that for all $x \in L^r_{p0}$,*

$$C_1 \| x^{(r)} \|_p \leq \| Ax \|_p \leq C_2 \| x^{(r)} \|_p \qquad (1 \leq p \leq \infty). \tag{3}$$

The second of these inequalities is obvious; the first follows from the second and the fact that L^r_{p0} is a Banach space in both norms $\| x^{(r)} \|_p$ and $\| Ax \|_p$.

An immediate consequence of this lemma is that at least the rough order of decrease of n-widths for W^r_p and $W_p(A)$ is the same (namely, $n^{-\alpha}$, $\alpha = \alpha(p, q, r) > 0$).

The next statement can be obtained by a simple geometric argument.

LEMMA 2. *Let*

$$\delta(M, \varepsilon) = \sup\{ \| x \|_\infty : \| x' \|_\infty \leq M, \| x \|_1 \leq \varepsilon \}.$$

Then $\delta(M, \varepsilon) \to 0$ if $\varepsilon \to 0$ for any fixed $M > 0$.

LEMMA 3 [5]. *There exists such C_3 that for any $x \in \tilde{L}^r_p$ $(1 \leq p \leq \infty)$, $1 \leq k \leq r$,*

$$\| x^{(k)} \|_p \leq C_3 \| x \|_p^{(r-k)/r} \cdot \| x^{(r)} \|_p^{k/r}.$$

Before proceeding to the proof of the theorem we note the following important duality:

$$d_n\big(W_p(A), L_q\big) \sim d^n\big(W_{q'}(A'), L_{p'}\big), \tag{4}$$

where $p' = p \cdot (p-1)^{-1}$, $q' = q \cdot (q-1)^{-1}$ and A' is the adjoint differential operator.

To prove (4) it suffices to show that

$$d_n\left(\overset{0}{W}_p (A), L_q \right) = d^n\left(\overset{1}{W}_{q'} (A'), L_{p'} \right). \tag{4'}$$

Indeed, if $x \in W_p(A)$ then $x = x_0 + z$, where $x_0 \in \overset{0}{W}_p(A)$, $Az = 0$. So $d_m(\overset{0}{W}_p(A), L_q) = d_{m+r}(W_p(A), L_q)$. A similar argument can be applied to $W_{q'}(A')$.

Let Γ_n be an arbitrary n-dimensional subspace of L_q. Then

$$\sup_{x \in \overset{0}{W}_p (A)} \inf_{\gamma \in \Gamma_n} \| x - \gamma \| = \sup \langle x, y \rangle \tag{5}$$

where the last supremum is taken over all $x \in \overset{0}{W}_p(A)$, $y \in L_{p'}$, $\| y \|_{p'} \leq 1$, $y \perp \Gamma_n$.

We define u by

$$A'u = y, \qquad u^{(k)}(1) = 0 \qquad (k = 0, 1, \dots, r-1). \tag{6}$$

Then $\langle x, y\rangle = \langle x, A'u\rangle = \langle Ax, u\rangle$ and

$$\sup_{x \in \overset{0}{W}_p(A)} \langle x, y\rangle = \sup_{x \in \overset{0}{W}_p(A)} \langle Ax, u\rangle = \sup_{\|\zeta\|_p \leqslant 1} \langle \zeta, u\rangle = \|u\|_{p'}. \tag{7}$$

It follows from (6) that $u \in \overset{1}{W}_{p'}(A')$. The condition $y \perp \Gamma_n$ means, in terms of u, that u belongs to some subspace of codimension n. So, by taking the infimum over Γ_n in (5) and (7) we have

$$d_n\big(W_p(A), L_q\big) \geqslant d^n\big(W_{q'}(A'), L_{p'}\big).$$

The opposite inequality can be obtained by inverting the preceding argument.

It follows from (4) that d_n in (2) can be replaced by d^n. On the other hand, $\tilde{W}_p$ is defined by imposing $2r$ linear conditions on $W_p(A)$. This produces at most $2r$-units shift of the index n in d^n which does not affect the asymptotic behavior of d^n since its order is $n^{-\alpha}$. So in order to prove the theorem it suffices to prove that

$$d^n\big(\tilde{W}_p(A), L_q\big) \sim d^n\big(\tilde{W}_p^r, L_q\big). \tag{8}$$

Now, for a subspace Γ^n (codim $\Gamma^n = n$), let

$$\sup_{x \in \tilde{W}_p^r \cap \Gamma^n} \|x\|_q = \varepsilon. \tag{9}$$

Suppose $x \in \tilde{W}_p(A) \cap \Gamma^n$. We can assume, without loss of generality, that $\|Ax\|_p = 1$. On the other hand, by Lemma 1 we have $\|x^{(r)}\|_p \leqslant C_1^{-1}$, so (9) implies that $\|x\|_q \leqslant C_1^{-1}\varepsilon_n$. Since $x \in \tilde{L}_p^r$ we also have $\|x'\|_\infty \leqslant C_1^{-1}$,

$$\|x\|_p \leqslant \|x\|_\infty \leqslant \delta\big(C_1^{-1}, C_1^{-1}\varepsilon\big) = \delta \tag{10}$$

and, by Lemma 2, $\delta \to 0$ when $\varepsilon \to 0$. Combining this fact with Lemma 3 we have $\|x^{(k)}\|_p \to 0$ when $\varepsilon \to 0$. So

$$\eta = \sup_{x \in \tilde{W}_p(A)} \|Ax - x^{(r)}\|_p \to 0$$

if $\varepsilon \to 0$. Since $x \in (1+\eta)\cdot \tilde{W}_p^r \cap \Gamma^n$ we have $\|x\|_q \leqslant \varepsilon(1+\eta)$. So if the subspace Γ^n is 'good' (in terms of d^n) for $\tilde{W}_p^r$ it is asymptotically equally good for $\tilde{W}_p(A)$.

To prove the opposite statement suppose that

$$\sup_{x \in \tilde{W}_p(A) \cap \Gamma^n} \|x\|_q = \varepsilon$$

and let $x \in \tilde{W}_p^r \cap \Gamma^n$. Then, assuming that $\|x^{(r)}\|_p = 1$, we have from Lemma 1 that $\|Ax\|_p \leqslant C_2$. So, by a similar argument, $\|x\|_q \leqslant C_2\varepsilon$, $\|x\|_p \leqslant \delta(1, C_2\varepsilon)$ and the proof can be concluded as above.

References

1. Charles K. Chui and Phillip W. Smith, *Some nonlinear spline approximation problems related to N-widths*, J. Approx. Theory **13** (1975), 421–430.

2. C. A. Micchelli and A. Pinkus, *On n-widths in L^∞: Limit as $n \to \infty$*, IBM Research Report RC 5573, 1975.

3. ______, *On n-widths in L^∞*, Trans. Amer. Math. Soc. **234** (1977), 139–174.

4. K. Nasyrova, *On asymptotic behavior of certain compacts in the space* $L_2[0, 1]$, Mat. Zametki **20** (3) (1973), 331–339. (Russian)

5. E. M. Stein, *Functions of exponential type*, Ann. of Math. (2) **65** (1957), 582–592.

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