ON THE HYPERBOLIC RIESZ MEANS

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Abstract. We define the hyperbolic Riesz means in \( \mathbb{R}^2 \) by \( H_\lambda f = (m_\lambda f) \) where 
\[ m_\lambda(\xi_1, \xi_2) = (1 - (\xi_1 \xi_2)^2)^\lambda, \lambda \geq 0, \]
and show that \( H_\lambda \) is bounded on \( L^p(\mathbb{R}^2) \) for \( \frac{1}{2} \leq p \leq 4 \) and \( \lambda > \frac{1}{2} \) or \( 1 < p < \infty \) and \( \lambda > 1 \).

Introduction. Several authors [1, 3] have studied the Riesz means for various domains in the plane, using the euclidean distance to the boundary of these domains. In the multipliers \( m_\lambda \), we use the hyperbolic distance to the characteristic set: \( \xi_1 \xi_2 = 0 \). Related results will appear in a separate paper.

1. We first estimate the Fourier transform of \( m_\lambda \). A simple calculation gives
\[
\hat{m}_\lambda(x, y) = \int_0^\infty \cos \left( \frac{x y}{u} \right) J_{\lambda + 1/2}(u) \frac{du}{u^{3/2 + \lambda}}.
\]

Lemma 1. Let
\[
h_\lambda(t) = \int_0^\infty \cos \left( \frac{t}{u} \right) J_{\lambda + 1/2}(u) \frac{du}{u^{3/2 + \lambda}};
\]
then:

1. \( h_\lambda(t) = O(\log |t|) \) for \( t \sim 0 \);
2. \( h_\lambda(t) = O(1/|t|^{3/4 + \lambda/2}) \) for \( |t| \sim \infty \).

Proof. This is a consequence, after some simple estimates, of the well-known estimates on Bessel functions.

Remark. From Lemma 1, we deduce the following consequence: let
\[
K_\lambda(x, y) = \frac{1}{\Gamma(\lambda + 1)} (1 - (xy)^2)^\lambda_+.
\]
Then the operator
\[
K_\lambda f(x, y) = \int_{\mathbb{R}^2} f(x - s, y - t) K_\lambda(s, t) \, ds \, dt
\]
is not bounded on \( L^2(\mathbb{R}^2) \) for any \( \lambda \geq 0 \).

For a function \( \phi \) of one variable \( t \) we let \( \eta_+(y) = \int_0^\infty \phi(t) t^{-i y} \, dt \) and \( \eta_-(y) = \int_0^\infty \phi(-t) t^{-i y} \, dt \).
**Lemma 2.** If a measurable function $\phi(t)$ satisfies: (1) $\int_0^\infty \phi(t) \, dt = 0$, (2) $\eta_+$ and $\eta_-$ are $C^1$ near $\gamma = 0$, and (3) $|\eta_+(\gamma)| + |\eta_-(\gamma)| \leq C/(1 + |\gamma|)$, then the operator

$$Tf(x, y) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x - s, y - t) \phi(st) \, ds \, dt$$

is bounded on $L^p(\mathbb{R}^2)$ for all $1 < p < \infty$.

**Proof.** We write

$$Tf(x, y) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x - s, y - t) \phi(st) \frac{ds}{|s|} \frac{dt}{|t|}$$

with $\psi(v) = |v| \phi(v)$. We then use the Mellin transform to write

$$\int_0^\infty \psi(v) v^{-i\gamma} \frac{dv}{v} = \int_0^\infty \phi(v) v^{-i\gamma} dv = \eta_+(\gamma),$$

$$\int_0^\infty \psi(-v) v^{-i\gamma} \frac{dv}{v} = \int_0^\infty \phi(-v) v^{-i\gamma} dv = \eta_-(\gamma).$$

Then

$$Tf(x, y) = \int_{s \geq 0} + \int_{s < 0} = 2 \int_{-\infty}^\infty (\eta_+(\gamma) + \eta_-(\gamma)) R_\gamma f(x, y) \, d\gamma$$

where

$$R_\gamma f(x, y) = \int_{-\infty}^\infty \int_{-\infty}^\infty f(x - s, y - t) \frac{ds}{|s|} \frac{dt}{|t|^{1-i\gamma}}.$$

We now let $M_\gamma f = (|\xi_1|^{-i\gamma} |\xi_2|^{-i\gamma})^\gamma$. Then $M_\gamma = C_\gamma^2 R_\gamma$ with

$$C_\gamma = \frac{\Gamma((1 + i\gamma)/2)}{\pi^{i\gamma + 1/2} \Gamma(-i\gamma/2)}.$$

From [2, Lemma (2.4)] we have

$$\|M_\gamma\|_{L^p} \leq (\text{const})(1 + |\gamma|^{21/p - 1/2} \log^2 |\gamma|)$$

for $1 < p < \infty$. We then write

$$Tf(x, y) = 2 \int_{-1}^1 (\eta_+(\gamma) + \eta_-(\gamma)) R_\gamma f(x, y) \, d\gamma + 2 \int_{|M| > 1} (\eta_+(\gamma) + \eta_-(\gamma)) R_\gamma f(x, y) \, d\gamma$$

$$= I + II.$$

For $I$, we have

$$I = 2 \int_{-1}^1 (\eta_+(\gamma) + \eta_-(\gamma)) C_\gamma^{-2} M_\gamma f(x, y) \, d\gamma.$$
Now, $$\eta_+(0) + \eta_-(0) = \int_{-\infty}^{\infty} \phi(t) \, dt = 0$$ by hypothesis (1). By hypothesis (2) $$\eta_+(\gamma) + \eta_-(\gamma) \sim (\text{const}) \gamma$$ for $$\gamma \to 0$$. If we use the fact that $$c_{\gamma} \sim \gamma$$ for $$\gamma \to 0$$, we may write

$$I = (\text{const}) \int_{-1}^{1} \gamma c_{\gamma}^{-2} M_{\gamma} f(x, y) \, d\gamma$$

$$= (\text{const}) \int_{-1}^{1} M_{\gamma} f(x, y) \, d\gamma + \text{bounded operator}$$

$$= (\text{const}) Uf(x, y) + \text{bounded operator}$$

where $$Uf(x, y) = (\sigma \hat{f})^\alpha(x, y)$$ with

$$\sigma(\xi_1, \xi_2) = \int_{0}^{\infty} \log |\xi_2| \sin u \, du.$$

The multiplier $$\sigma$$ is easily seen to verify $$|\xi^\alpha \partial_x^\alpha \sigma(\xi)| \leq C_\alpha$$ for all $$\alpha = (\alpha_1, \alpha_2)$$, which shows, using a well-known result [4, Theorem 6], that $$U$$ is bounded on $$L^p(\mathbb{R}^2)$$ for all $$1 < p < \infty$$.

For II we use hypothesis (3), the estimates on $$\|M_{\gamma}\|_{L^p}$$ given above and the fact that $$C_{\gamma} = O(\|\gamma\|)$$ as $$|\gamma| \to \infty$$, to get

$$\|\Pi\|_p \leq (\text{const}) \int_{|\gamma| > 1} \left( \frac{|\gamma|^{2(1/p - 1/2)}}{|\gamma|^2} \log^2 |\gamma| \right) d\gamma \|f\|_p.$$ 

The last integral is clearly convergent for all $$p > 1$$.

THEOREM. The hyperbolic Riesz mean $$H_\lambda$$ is bounded on $$L^p(\mathbb{R}^2)$$ for all $$\lambda > 0$$ and $$p > 1$$ such that $$\lambda > |2/p - 1|$$.

PROOF. From Lemma 1, we have $$H_\lambda f = h_\lambda, f$$ where

$$h_\lambda(t) = \int_{0}^{\infty} \cos \left( \frac{t}{u} \right) J_{\lambda+1/2}(u) \frac{du}{u^{3/2 + \lambda}}.$$ 

Condition (1) of Lemma 2 is satisfied since

$$\int_{-\infty}^{\infty} h_\lambda(t) \, dt = \frac{1}{2} J_{\lambda+1/2} \left( \frac{1}{u} \right) |u|^{\lambda-1/2} \bigg|_{u=0} = 0.$$ 

Condition (2) is easily verified since $$h_\lambda(t) = \log |t| + \phi(t)$$ for $$t \sim 0$$, where $$\phi$$ is a smooth function, and

$$h_\lambda(t) = \frac{\exp \left( \sqrt{t} \right)}{|t|^{3/4 + \lambda/2}} \psi(t)$$

for $$t \sim \infty$$, where $$\psi$$ is a smooth function which verifies $$|\partial_t^k \psi(t)| \leq C_k (1 + |t|)^{-k}$$, and the integrals

$$\int_{0}^{1} \log^2 t \, dt \quad \text{and} \quad \int_{1}^{\infty} \exp \left( \sqrt{t} \right) \log t \, dt$$

are convergent. For condition (3), a simple estimate shows that

$$|\eta_+ (\gamma)| + |\eta_- (\gamma)| \leq C / (1 + |\gamma|)^{\lambda}.$$
To conclude we need the integral
\[ \int_0^\infty \frac{|\gamma|^{21/p - 1/2}}{(1 + |\gamma|)^{\lambda+1}} \log^2(1 + |\gamma|) \, d\gamma \]
to be convergent, i.e., \( \lambda + 1 - \frac{2}{p} - 1 > 1 \).

**Corollary.** \( H_\lambda \) is bounded on \( L^p(\mathbb{R}^2) \) for (1) \( \lambda > \frac{1}{2} \) and \( \frac{4}{3} \leq p \leq 4 \); (2) \( \lambda \geq 1 \) and \( 1 < p < \infty \).

**Remark.** For the Riesz means in \( \mathbb{R}^2 \), it is known that the critical index is \( \frac{1}{2} \) \((= (n - 1)/2)\). For \( \lambda > \frac{1}{2} \), one has trivially both the \( L^p \)-estimates for \( 1 < p < \infty \) and the pointwise convergence associated with the Riesz means. In the case of the operators \( H_\lambda \), one needs to dissociate these two properties. For the \( L^p \)-estimates, \( 1 < p < \infty \), we conjecture that the critical index is \( 1 \) \((= n/2)\). This loss of \( \frac{1}{2} \) might occur because the hyperbolas have vanishing curvature at infinity. For the corresponding pointwise convergence, it is an open problem for any \( \lambda \geq 0 \).

**References**


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