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A NOTE ON JENSEN'S COVERING LEMMA

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ABSTRACT. We show that Jensen's covering lemma does not hold for order-types instead of cardinalities.

A famous result of Jensen says that if $0^{\#}$ does not exist then every uncountable set can be covered by a constructible set of the same power [1]. We show that an iterated forcing construction can produce a model such that for every $\alpha < \omega_2$ there is a set (in the enlarged model) of order-type ω_1 which cannot be covered by an old set of order-type $\leqslant \alpha$.

THEOREM. *Assume that V is a countable, transitive model of ZFC + GCH. Then there is a cardinal preserving generic extension $V[G]$ such that for every $\alpha < \omega_2$ there is an $X \subseteq \omega_2$, $\mathrm{tp}(X) = \omega_1$, such that a $Y \in V$ with $Y \geqslant X$, $\mathrm{tp}(Y) < \alpha_1$, does not exist.*

PROOF. The appropriate notion of forcing will be P_{ω_2} where $P_\alpha(\alpha \leqslant \omega_2)$ is defined by induction. P_0 is just the trivial notion of forcing. $P_{\alpha+1} = P_\alpha * Q_\alpha$, where Q_α will be defined later; for limit α we take inverse limits when $\mathrm{cf}(\alpha) = \omega$ and direct limits if $\mathrm{cf}(\alpha) > \omega$.

We define Q_α (inside V^{P_α}) as follows: $(f, A) \in Q_\alpha$ if and only if

(i) $\mathrm{Dom}(f) \in \omega_1$,

(ii) $\omega_1^{\alpha+1}\xi \leqslant f(\xi) < \omega_1^{\alpha+1}(\xi + 1)$ $(\xi \in \mathrm{Dom}(f))$,

and

(iii) $\mathrm{tp}(A \cap [\omega_1^{\alpha+1}\xi, \omega_1^{\alpha+1}(\xi+1)]) < \omega_1^{\alpha+1}$.

The partial order is defined by $(f', A') \leqslant (f, A)$ iff $f' \supseteq f$, $A' \supseteq A$, and $f'(\xi) \notin A$ for $\xi \in \mathrm{Dom}(f') - \mathrm{Dom}(f)$. It is easy to see that this relation is transitive.

In order to prove that forcing with any of the $P_\alpha(\alpha \leqslant \omega_2)$ cardinals and cardinal arithmetic remain we need some tools elaborated by Baumgartner [2]. A partial order P is σ-closed if every decreasing sequence $p_0 \geqslant p_1 \geqslant \cdots \geqslant p_n \geqslant \cdots (n < \omega)$ has a lower bound. P is well-met iff every two compatible elements have a greatest lower bound. P is ω_1-linked iff it can be written as $\bigcup\{R_\tau\colon \tau < \omega_1\}$ where the elements in any of the R_τ's are pairwise compatible. Baumgartner proves that if $P_\alpha(\alpha \leqslant \omega_2)$ is defined as above and Q_α is (in V^{P_α}) σ-closed, well-met and ω_1-linked, then P_{ω_2} is σ-closed and has the ω_2-chain condition.

LEMMA 1. *The partial ordering Q_α is σ-closed, well-met and ω_1-linked.*

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PROOF. Assume (f_n, A_n) is decreasing. At least, $f = \bigcup f_n$, $A = \bigcup A_n$ seems to give a candidate. As $\omega_1^{\alpha+1}$ is indecomposable into countably many smaller ordinals, the (f, A) is a condition. Assume $\xi \in \mathrm{Dom}(f) - \mathrm{Dom}(f_n)$, and there is an m with $\xi \in \mathrm{Dom}(f_m + 1) - \mathrm{Dom}(f_m)$. Then $\xi \notin A_m \supseteq A_n$.

The ordering is ω_1-linked as (f, A) and (f, B) are always compatible and by CH there are only ω_1 first coordinates.

To prove that Q_α is well-met assume (f, A) and (g, B) are compatible and $\mathrm{Dom}(f) < \mathrm{Dom}(g)$. As some $(h, C) \leqslant (f, A), (g, B)$, so if $\xi \in \mathrm{Dom}(f) - \mathrm{Dom}(g)$ surely $\xi \in \mathrm{Dom}(h) - \mathrm{Dom}(f)$ so $g(\xi) = h(\xi) \notin A$. This gives that $(f \cup g, A \cup B)$ is the g.l.b. for $(f, A), (g, B)$.

We finish the proof by showing that the major conclusion of the theorem holds in V^P.

LEMMA 2. *Forcing with Q_α defines a set X_α of order-type ω_1 which cannot be covered by a set of order-type $\leqslant \omega_1^{\alpha+1}$ in the ground model.*

PROOF. Let G be a generic set over Q_α. Put $X_\alpha = \bigcup\{\mathrm{Rug}(f)\colon (f, A) \in G\}$. As $D_\xi = \{(f, A)\colon \xi \in \mathrm{Dom}(f)\}$ is clearly dense for $\xi < \omega_1$, the order-type of X_α is ω_1. Assume Y is the ground model $Y \subseteq \omega_1^{\alpha+2}$, $\mathrm{tp}\, Y \leqslant \omega_1^\alpha$ and $(f, A) \Vdash \underline{X}_\alpha \subseteq Y$. Then $(f, A \cup Y)$ is a condition forcing $\underline{X}_\alpha \cap Y \subseteq \mathrm{Rug}(f)$ which is countable, so $\underline{X}_\alpha \subseteq Y$ cannot hold.

REFERENCES

1. K. Devlin and R. B. Jensen, *Marginalia to a theorem of Silver*, Logic Conference, Kiel 1974, Lecture Notes in Math., vol. 499, Springer-Verlag, Berlin, 1975, pp. 115–142.

2. J. E. Baumgartner, *Iterated forcing*, Proc. Cambridge Summer School on Set Theory, 1978 (to appear).

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