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ORDER OF MAGNITUDE OF THE CONCENTRATION FUNCTION

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ABSTRACT. Suppose a sum of independent random variables, when scaled in a suitable way, is stochastically compact. It is proved that the precise order of magnitude of the concentration function of the sum equals the inverse of the scale factor.

1. Introduction and results. Let X, X_1, $X_2, \ldots$ be independent and identically distributed random variables, and set $S_n = \Sigma_1^n X_j$. Weak limit theorems for S_n usually take the form

$$Y_n \equiv (S_n - \text{med } S_n)/b_n \to Z$$

in distribution, where med S_n denotes a median of S_n, $\{b_n\}$ is a sequence of norming constants and Z is a nondegenerate limit. Feller **[2]** showed that interesting and useful results can still be obtained if the condition of convergence in distribution is weakened to that of stochastic compactness. The sequence $\{Y_n\}$ is stochastically compact if every subsequence contains a further subsequence converging in distribution to a proper, nondegenerate limit. We shall show in this paper that the condition of stochastic compactness permits a simple description of the precise order of magnitude of the concentration function.

The existence of a sequence $\{b_n\}$ such that $\{Y_n\}$ is stochastically compact, is equivalent to the condition

$$\limsup_{x \to \infty} x^2 P(|X| > x)/E\{X^2 I(|X| \leq x)\} < \infty \tag{1.1}$$

[2, p. 387]. Such constraints are related to characterisations of domains of partial attraction; see Jain and Orey **[5]** and Maller **[6]**. An alternative definition of stochastic compactness may be given in the following way. Define the function v by

$$\begin{aligned} v(x) &= x^{-2} \int_0^x u P(|X| > u)\, du \\ &= \tfrac{1}{2}\left[x^{-2} E\{X^2 I(|X| \leq x)\} + P(|X| > x)\right], \qquad x > 0. \end{aligned}$$

As was remarked in **[4]**, v is continuous, v' exists and is negative at continuity points of $|X|$, and v is ultimately strictly decreasing. Hence the equation $v(\eta) = x^{-1}$ admits a unique solution, $\eta = \eta(x)$, for large x.

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LEMMA. *Condition* (1.1) *is equivalent to the constraint that* η *varies dominatedly at infinity. When either of these conditions holds and we take* $b_n = \eta(n)$ *in the definition of* Y_n, *the sequence* $\{Y_n\}$ *is stochastically compact.*

The condition of dominated variation (equivalent here to R-O variation) is discussed at length in the Appendix to Seneta [**8**]. In the present situation, dominated variation means that for some $\lambda > 1$, $\eta(\lambda x)/\eta(x)$ is bounded as $x \to \infty$. We shall prove that when η varies dominatedly, the sequence $\{1/\eta(n)\}$ is of the same order of magnitude as the concentration function,

$$Q(S_n; h) = \sup_x P(x < S_n \leqslant x + h).$$

Therefore we may take $b_n = 1/Q(S_n; h)$ in the definition of Y_n.

THEOREM. *If* η *varies dominatedly then for each* $h > 0$,

$$0 < \liminf_{n\to\infty} Q(S_n; h)\eta(n) \leqslant \limsup_{n\to\infty} Q(S_n; h)\eta(n) < \infty. \tag{1.2}$$

This result greatly extends earlier estimates due to Esséen [**1**], who assumed X to be in the domain of attraction of a stable law. See Petrov [**7**, Chapter III] for a review of the properties of concentration functions.

Since the preparation of this paper and its first revision, the author has seen a preprint by Griffin, Jain and Pruitt [**3**] which describes results which overlap with (but do not contain) those presented here.

2. Proofs.

PROOF OF LEMMA. We prove first that the dominated variation of η implies (1.1). If (1.1) fails then it fails along a subsequence $x_k \uparrow \infty$, which entails

$$v(x_k) \sim \tfrac{1}{2}P(|X| > x_k) \tag{2.1}$$

as $k \to \infty$. Whenever $0 < \rho \leqslant 1$ we have

$$v(x) \geqslant x^{-2}\int_0^{\rho x} uP(|X| > u)\,du + x^{-2}P(|X| > x)\int_{\rho x}^{x} u\,du$$

$$= \rho^2 v(\rho x) + \tfrac{1}{2}(1 - \rho^2)P(|X| > x).$$

The right-hand side dominates $\frac{1}{2}P(|X| > x)$ (it is an increasing function of ρ, and so we let $\rho \to 0$). Therefore it follows from (2.1) that $v(\rho x_k) \sim v(x_k)$ as $k \to \infty$. Let $\rho x_k = \eta(y_k)$, where y_k depends on ρ, and observe that

$$y_k^{-1} = v(\eta(y_k)) = v(\rho x_k) \sim v(x_k) = v(\rho^{-1}\eta(y_k)).$$

Suppose $\lambda > 1$. If $\eta(\lambda y_k) \leqslant \rho^{-1}\eta(y_k)$ for all large k, then

$$(\lambda y_k)^{-1} = v(\eta(\lambda y_k)) \geqslant v(\rho^{-1}\eta(y_k)) \sim y_k^{-1},$$

which is impossible. Consequently $\limsup_{x\to\infty} \eta(\lambda x)/\eta(x) \geqslant \rho^{-1}$, and since this is true for all $\lambda > 1$ and $0 < \rho \leqslant 1$, η cannot vary dominatedly. This contradiction proves (1.1).

Conversely, if (1.1) holds then the sequence $\{Y_n\}$ is stochastically compact whenever b_n/a_n is bounded away from zero and infinity, where a_n is given by

$$na_n^{-2}E\{X^2 I(|X| \leqslant a_n)\} = 1;$$

see [**2**, p. 387]. It is not difficult to show that for each $c > 0$, $b_n = \eta(cn)$ satisfies this condition. (If b_n satisfies the condition, so does $b'_n = b_{[cn]}$. It is readily checked that under (1.1), $b_n = \eta(n)$ satisfies the condition.) Therefore $\eta(cn)/\eta(n)$ is bounded away from zero and infinity as $n \to \infty$, for each $c > 0$, and so (since η is monotone) η varies dominatedly.

PROOF OF THEOREM. The symbol C below denotes a generic positive constant. Let $\eta_s(n)$ ($= \eta_s$ for short) and v_s denote the versions of $\eta(n)$ and v for the symmetrisation, $\tilde{X}$, of X. It was proved in [**4**] that $\eta(x) \leqslant \eta_s(2x) \leqslant 2\eta(4x)$,[1] and from this it may be proved that η varies dominatedly iff η_s varies dominatedly.

Choose n so large that $1 - |\phi(t)|^2 \leqslant \frac{1}{2}$ for $0 \leqslant t \leqslant 1/\eta_s$, where ϕ denotes the characteristic function of X. By Esséen's [**1**] Main Lemma,

$$\begin{aligned} CQ(S_n; h) &\geqslant \int_0^{1/\eta_s} |\phi(t)|^{2n}\, dt \geqslant \eta_s^{-1}\int_0^1 \exp\left[-2nE\left\{1 - \cos\left(t\tilde{X}/\eta_s\right)\right\}\right] dt \\ &\geqslant \eta_s^{-1}\int_0^1 \exp\left(-4n\left[E\left\{\left(t\tilde{X}/\eta_s\right)^2 I\left(|\tilde{X}| \leqslant \eta_s\right)\right\} + P\left(|\tilde{X}| > \eta_s\right)\right]\right) dt \\ &\geqslant \eta_s^{-1}\int_0^1 \exp\{-8nv_s(\eta_s)\}\, dt = \eta_s^{-1}e^{-8}. \end{aligned}$$

Since $\eta_s(n) \leqslant 2\eta(2n)$ then $Q(S_n; h) \geqslant C/\eta(2n)$, and the left-hand inequality in (1.2) follows from the dominated variation of η.

By the Main Lemma of [**1**] we have for each $\varepsilon > 0$,

$$\begin{aligned} C(\varepsilon)Q(S_n; h) &\leqslant \int_0^{\varepsilon} |\phi(t)|^n\, dt = \eta_s^{-1}\int_0^{\varepsilon\eta_s} |\phi(t/\eta_s)|^n\, dt \\ &\leqslant \eta_s^{-1}\int_0^{\varepsilon\eta_s} \exp\left[-(n/2)E\left\{1 - \cos\left(t\tilde{X}/\eta_s\right)\right\}\right] dt. \end{aligned}$$

Since

$$\begin{aligned} E\{1 - \cos(t\tilde{X})\} &\geqslant E\left[\{1 - \cos(t\tilde{X})\}I\left(|\tilde{X}| \leqslant t^{-1}\right)\right] \\ &\geqslant C_1 t^2 E\left\{\tilde{X}^2 I\left(|\tilde{X}| \leqslant t^{-1}\right)\right\} \geqslant C_2 v_s(t^{-1}) \end{aligned}$$

for small t, using (1.1), then if ε is sufficiently small,

$$Q(S_n; h) \leqslant C_4\eta_s^{-1}\int_0^{\varepsilon\eta_s} \exp\{-C_3 n v_s(\eta_s/t)\}\, dt. \tag{2.2}$$

Since η_s is R-O varying, it may be deduced from Theorem A.1, p. 93 of [**8**] that for some $c \geqslant 1$ and $k \geqslant 1$, $\eta_s(xy) \leqslant \eta_s(x)y^c$ whenever $x \geqslant k$ and $y \geqslant k$. Therefore

$$\eta_s(x)/t = \eta_s\left(xt^{-1/c}t^{1/c}\right)/\left(t^{1/c}\right)^c \leqslant \eta_s\left(xt^{-1/c}\right)$$

and

$$v_s\left(\eta_s(x)/t\right) \geqslant v_s\left(\eta_s\left(xt^{-1/c}\right)\right) = t^{1/c}/x,$$

whenever $t \geqslant k^c$ and $x \geqslant kt^{1/c}$. Thus, provided $n \geqslant k\{\varepsilon\eta_s(n)\}^{1/c}$,

$$nv_s\left(\eta_s(n)/t\right) \geqslant t^{1/c}, \tag{2.3}$$

[1] Both inequalities are proved in [**4**], but only the first is used. In [**4**, p. 567, line 7], "first" should read "second".

whenever $k^c \leqslant t \leqslant \varepsilon\eta_s(n)$. It also follows from Theorem A.1 of [**8**] that for some $d > 0$, $\eta_s(n) \leqslant n^d$ when n is large. If we choose $c > d$ then (2.3) certainly holds for large n, and thus

$$\int_{k^c}^{\varepsilon\eta_s} \exp\{-C_3 n v_s(\eta_s/t)\}\, dt \leqslant \int_0^\infty \exp(-C_3 t^{1/c})\, dt < \infty.$$

The right-hand inequality in (1.2) now follows from (2.2) and the fact that $\eta_s(n) \geqslant \eta(n/2)$.

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