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ON THE ALEXANDER POLYNOMIAL OF A CYCLICALLY PERIODIC KNOT

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ABSTRACT. We show that a theorem of Burde on the Alexander polynomial of a cyclically periodic knot K, such that $\Delta_2(K) = 1$, may be extended to all cyclically periodic knots.

A knot K in a homology 3-sphere Σ has cyclic period q if it is invariant under an automorphism of Σ of order q which has fixed point set a circle disjoint from K. In [**3**] we gave new proofs of theorems of Burde [**1**] and Murasugi [**5**] on the Alexander polynomial $\Delta_1(K)$ of such a knot.

A refinement of the second lemma of [**3**] enables us to remove the restriction $\Delta_2(K) = 1$ from Burde's theorem. We shall adopt the notation of [**3**].

THEOREM. *Let K be a knot in an homology 3-sphere Σ which has cyclic period q, where $q = p^r$ for some prime p. Then either*

(i) $\Delta_1(K) \equiv 1$ modulo (p) *or*

(ii) *the degree of the qth roots of unity over the splitting fields of $\Delta_1(K)$ is at most n, where $\Delta_n(K) \neq 1$ but $\Delta_{n+1}(K) = 1$.*

PROOF. The argument given in [**3**] shows that if (i) fails then the $Q[t, t^{-1}]$-module $C = (G'/G'') \otimes_{\mathbf{Z}} Q$ (the rational homology of the infinite cyclic cover of the knot complement) has an automorphism of order exactly q. (For otherwise the (q/p)th power of the given periodic self-homeomorphism τ of Σ induces the identity on G'/G''. Applying Milnor duality and the Wang sequence with coefficients $\mathbf{Z}/p\mathbf{Z}$ to the p-fold branched cyclic covering map from Z to $\bar{Z}$, the results of 0-framed surgery on K in Σ and on the image of K in $\Sigma/\tau^{(q/p)}$, respectively, we conclude that $(G'/G'') \otimes (\mathbf{Z}/p\mathbf{Z})$ must be 0, and so $\Delta_1(K) \equiv 1$ modulo (p).) Then for some irreducible factor δ of $\Delta_1(K)$, the δ-primary submodule of C has an automorphism τ of order exactly q, so we may assume $\Delta_0(C) = \delta^l$. Let $L = Q[t, t^{-1}]/(\delta)$. If $m \geqslant 1$ the kernel of the natural map from $\mathrm{Aut}(C/\delta^{m+1}C)$ to $\mathrm{Aut}(C/\delta^m C)$ is isomorphic to $\mathrm{Hom}(C/\delta^{m+1}C, \delta^m C/\delta^{m+1}C)$ and so is an L-vector space. In particular, it is torsion free. Therefore the image of τ in $\mathrm{Aut}(C/\delta C) \approx \mathrm{GL}(n, L)$ has order exactly q. Let $A = L[\tau]$ be the subalgebra of $M(n, L)$ generated by this element. Then $\dim_L A \leqslant n$ by the Cayley-Hamilton theorem. On the other hand, the image of τ in the semisimple algebra $A/\mathrm{rad}\, A$ again has order exactly q, as the kernel of the map on

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units is again an iterated extension of L-vector spaces. The algebra $A/\text{rad}\, A$ is a product of fields containing L, so some factor is an extension L' of L which contains the qth roots of unity and for which $[L' : L] = \dim_L L' \leqslant \dim_L A \leqslant n$. Since L is contained in the splitting field for $\Delta_1(K)$, the theorem follows.

REMARKS. Suppose (ii) holds. If $q = p^r$, with $r > 1$, then either $L = L'$, and so contains the qth roots of unity, or $[L' : L] \geqslant p$, and so $p \leqslant n$. If $q = p$ and L is linearly disjoint from the field $Q(\zeta)$ of pth roots of unity, that is if $L \cap Q(\zeta) = Q$, then $[L' : L] = p - 1$, so $p \leqslant n + 1$.

If $\Delta_2(K) = 1$, so that $n = 1$, we obtain Burde's theorem [**1**] (itself an extension of a result of Trotter [**8**]). Burde's version applies to all but four of the prime knots with ten crossings or less (excepting $8_{18}, 9_{40}, 10_{99}$ and 10_{123}), but does not usefully apply to the n-fold connected sum of a knot K with itself, for $\Delta_2(\#^n K)$ is nontrivial unless $\Delta_1(K) = 1$ or $n = 1$.

For other recent work on knots with cyclic periods see [**2, 3, 4, 6, 7**]. (Note that there are some discrepancies between the tables of [**4**] and [**6**].)

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