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SHORTER NOTES

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A NOTE ON THE AUTOMORPHISM TOWER THEOREM FOR FINITE GROUPS

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If G is a group with trivial center, the automorphism tower of G is the series $G = A_0 \lhd A_1 \lhd A_2 \lhd \cdots$, where A_{i+1} is the automorphism group of A_i (and A_i is identified with its inner automorphisms). It is a theorem of Wielandt [**5**] that if G is finite, its automorphism tower has finite height. In fact, slightly more generally, Wielandt showed that if G is a finite subnormal subgroup of A and $C_A(G) = 1$, then $|A|$ is bounded in terms of $|G|$. The proof of this theorem has been simplified in a number of ways over the years but the bounds obtained for $|A|$ in even the more recent literature (for example [**2**, **4**, and **7**]) are rather crude and awkward to state explicitly. The aim of this note is to record a very simple argument which yields a much more reasonable bound. There are no new ideas; only more effective exploitation of old ones.

The following notation will be used: $G \operatorname{sn} A$ means G is a subnormal subgroup of A; $\operatorname{Aut}(G)$, $Z(G)$ and $F(G)$ denote, respectively, the automorphism group of G, the center, and the Fitting subgroup of G; $C_A(G)$ denotes the centralizer of G in A; G^∞ is the nilpotent residual of G, the smallest normal subgroup of G with G/G^∞ nilpotent; $F^*(A)$ is the generalized Fitting subgroup introduced by Bender (so $F^*(A)/F(A)$ is generated by the minimal normal subgroups of $F(A)C_A(F(A))/F(A)$). All groups are assumed to be finite.

The argument depends on three basic facts, two of which have been used previously in the context of the tower theorem. All have quite short and elementary proofs.

(1) (Schenkman [**3**]) If $G \operatorname{sn} A$ with $C_A(G) = 1$, then $C_A(G^\infty) \leqslant G^\infty$.

(2) (Wielandt) If $B \operatorname{sn} A$ and B is simple nonabelian, then B normalizes every subnormal subgroup of A. (See, for example, 13.3.2 of [**2**].)

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(3) (Bender) For any finite group A, $C_A(F^*(A)) \leqslant F^*(A)$. (See, for example, Lemma 10.1 of [**1**].)

Wielandt proved in [**6**] that if G and H are subnormal in A, then $\langle G, H\rangle^\infty = G^\infty H^\infty$. Only the following special case is required here.

LEMMA 1. *Suppose F is a nilpotent normal subgroup of A. If G sn A, then $(FG)^\infty = G^\infty$.*

PROOF. G sn FG so we may induct on the minimal length of a subnormal series from G to FG. If $G \lhd H$ sn FG, then $FG = FH$ so by induction, $(FG)^\infty = (FH)^\infty = H^\infty$. On the other hand, $H = FG \cap H = (F \cap H)G$ and $G^\infty \lhd H$ (since $G \lhd H$) so $H/G^\infty = ((F \cap H)G^\infty/G^\infty)(G/G^\infty)$ which is clearly nilpotent. Thus, $H^\infty = G^\infty$ and the proof is complete.

LEMMA 2. *If G sn A, then $F^*(A)$ normalizes G^∞.*

PROOF. Let $N/F(A)$ be a simple subnormal subgroup of $F(A)C_A(F(A))/F(A)$. N is not solvable (since $N = F(A)C_N(F(A))$ whence $C_N(F(N)) \nleqslant F(N)$) so $N/F(A)$ is not abelian. By (2) applied to $A/F(A)$, N normalizes $F(A)G$. But by Lemma 1, $G^\infty = (F(A)G)^\infty$, a characteristic subgroup of $F(A)G$. Hence N normalizes G^∞ and the result follows from the definition of $F^*(A)$.

The proof of Wielandt's theorem is now easy.

THEOREM. *Suppose G sn A such that $C_A(G) = 1$. Then*

$$|A| \leqslant (|Z(G^\infty)|\,|\mathrm{Aut}(G^\infty)|)!.$$

PROOF. By Lemma 2 and (1) it follows that $|F^*(A)| \leqslant |Z(G^\infty)|\,|\mathrm{Aut}(G^\infty)|$. On the other hand, by (3), $|A| \leqslant |Z(F^*(A))|\,|\mathrm{Aut}(F^*(A))| \leqslant |F^*(A)|\,!$.

The tower theorem follows immediately from 13.5.3 of [**2**] and the preceding theorem.

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