PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 2, October 1983

SUPPORT POINTS OF THE UNIT BALL OF H^p $(1 \leqslant p \leqslant \infty)$

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ABSTRACT. The following results are obtained for the H^p class, over the open unit disc, whenever $1 \leqslant p \leqslant \infty$.

(1) f is a support point of the unit ball of H^p, whenever $1 \leqslant p < \infty$, if and only if $\|f\|_p = 1$ and f is of the form $f(z) = [Q(z)]^{2/p} \cdot W(z)$ where $W(z)$ is a function analytic in the closed unit disc and nonvanishing on its boundary and $Q(z)$ is either a nonzero constant or a polynomial with all of its zeros on the boundary of the unit disc.

(2) f is a support point of the unit ball of H^∞ if and only if f is a finite Blaschke product.

1. Introduction. Let $U = \{z: |z| < 1\}$, $\bar{U} = \{z: |z| \leqslant 1\}$ and $\partial U = \{z: |z| = 1\}$. Denote by A the space of functions analytic in U with the topology of uniform convergence on compact subsets of U. Each continuous linear functional L on A is given by a function

$$K(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^{n+1}} \tag{1}$$

analytic in $|z| > r_0$, for some $r_0 < 1$, with $\overline{\lim}_{n\to\infty} |b_n|^{1/n} < 1$ and so that

$$L(f) = \sum_{n=0}^{\infty} a_n b_n = \frac{1}{2\pi i} \int_{\substack{|z|=R \\ r_0<R<1}} f(z)K(z)\,dz \tag{2}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A$ [**5**, p. 36].

A function f in a compact subset F of A is called a support point of F if there is a continuous linear functional L on A, with $\operatorname{Re} L$ nonconstant on F, so that $\operatorname{Re} L(f) = \max_{g\in F} \operatorname{Re} L(g)$.

A function $f \in A$ is said to belong to the class H^p $(0 < p < \infty)$ if

$$\|f\|_p = \lim_{r\to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{1/p} < \infty.$$

The class of bounded analytic functions is denoted by H^∞ and $\|f\|_\infty = \lim_{r\to 1} \max_{0\leqslant\theta<2\pi} |f(re^{i\theta})|$. Each $f \in H^p$ has a radial limit $f(e^{i\theta})$ almost everywhere and $f(e^{i\theta}) \in L^p$.

Received by the editors December 9, 1982.

1980 *Mathematics Subject Classification*. Primary 30D55, 30D50.

Key words and phrases. Analytic function, continuous linear functional, H^p classes, support point, extreme point.

In a recent paper, D. J. Hallenbeck and T. H. MacGregor [2] showed that the set of support points of the unit ball of H^∞ consists of all finite Blaschke products, that is, functions of the form

$$\prod_1^n \frac{z+\alpha_k}{1+\bar{\alpha}_k z} \tag{3}$$

where $|\alpha_k| \leq 1$.

They also showed [2], by using the Cauchy-Schwarz inequality, that the set of support points of the unit ball of H^2 consists of all functions $f \in H^2$ that satisfy $\|f\|_2 = 1$ and analytic in $\bar{U}$. This led them [2] to ask the question of whether a similar result holds for the unit ball of H^p $(1 \leq p < \infty)$.

In §2, we determine that

(1) for $1 \leq p < \infty$, f is a support point of the unit ball of H^p if and only if $\|f\|_p = 1$ and f is of the form

$$f(z) = [Q(z)]^{2/p} \cdot W(z) \tag{4}$$

where W is a function analytic in $\bar{U}$ and nonvanishing on ∂U and $Q(z)$ is either a nonzero constant or a polynomial with all of its zeros on ∂U. Furthermore, by a method different than the one in [2], we determine that

(2) for $p = \infty$, f is a support point of the unit ball of H^∞ if and only if f is a finite Blaschke product.

2. Support points of the unit ball of H^p $(1 \leq p \leq \infty)$.

LEMMA 1. *Let $f \in H^p$ and $g \in H^q$, where $p \geq 1$ and $1/p + 1/q = 1$. Let K be a function analytic in $|z| > r_0$, for some $r_0 < 1$, and $zf(z)(K(z) - g(z)) = R(z)$. Then*

$$\lim_{r\to 1}\int_0^{2\pi} |rf(re^{i\theta})K(re^{i\theta}) - f(e^{i\theta})K(e^{i\theta})|\,d\theta = 0 \tag{5}$$

and

$$\lim_{r\to 1}\int_0^{2\pi} |R(re^{i\theta}) - R(e^{i\theta})|\,d\theta = 0. \tag{6}$$

PROOF. We show (5) first and then (6) follows from (5) and the fact that $zf \cdot g \in H^1$ [1, p. 21]. To show (5) write

$$\begin{aligned} rf(re^{i\theta})K(re^{i\theta}) - f(e^{i\theta})K(e^{i\theta}) \\ = rf(re^{i\theta}) + \big(K(re^{i\theta}) - K(e^{i\theta})\big) + K(e^{i\theta})\big(rf(re^{i\theta}) - f(e^{i\theta})\big). \end{aligned} \tag{7}$$

Since $K(z)$ is analytic near ∂U, thus uniformly continuous, we conclude that for any $\varepsilon > 0$ there is $1 > r_1 > r_0$ such that $|K(re^{i\theta}) - K(e^{i\theta})| < \varepsilon$, for all θ and all $r_1 < r \leq 1$, and, furthermore, $|K(e^{i\theta})| \leq M$ for all θ. This, (7) and the fact that $zf \in H^1$ imply statement (5).

Let L be a continuous linear functional on A defined as in (2). It follows immediately, from (5), that, whenever $f \in H^1$,

$$\lim_{r\to 1}\int_{|z|=r} f(z)K(z)\,dz = \int_{|z|=1} f(z)K(z)\,dz.$$

Hence (2) can be rewritten,

$$L(f) = \sum_{n=0}^{\infty} a_n b_n = \frac{1}{2\pi i} \int_{|z|=1} f(z) K(z)\, dz \tag{8}$$

whenever $f \in H^1$. Since $K(e^{i\theta})$ is continuous, it follows that [**1**, p. 132; **3**, p. 134] there is a function $f \in H^p$ $(p \geq 1)$ with $\|f\|_p = 1$ and a unique $g \in H^q$ $(1/p + 1/q = 1)$ so that

$$|L(f)| = \max\{|L(h)|: h \in H^p, \|h\|_p \leq 1\} = \|K - g\|_q = \inf_{h \in H^q} \|K - h\|_q. \tag{9}$$

If $p > 1$, there is a unique f with the normalization $L(f) > 0$. Furthermore, in order that f (with $Lf > 0$) and g satisfy (9), it is necessary and sufficient that [**1**, p. 133]

$$e^{i\theta} f(e^{i\theta})\left(K(e^{i\theta}) - g(e^{i\theta})\right) \geq 0 \tag{10}$$

for almost all θ, and that

$$|K(e^{i\theta}) - g(e^{i\theta})| = \|K - g\|_\infty \quad \text{for almost all } \theta, \text{ if } p = 1, \tag{11}$$

$$|f(e^{i\theta})|^p = \frac{|K(e^{i\theta}) - g(e^{i\theta})|^q}{\|K - g\|_q^q} \quad \text{for almost all } \theta, \text{ if } 1 < p < \infty, \tag{12}$$

$$|f(e^{i\theta})| = 1 \quad \text{almost everywhere on } \{\theta: K(e^{i\theta}) \neq g(e^{i\theta})\}, \text{ if } p = \infty. \tag{13}$$

LEMMA 2. *Let $f \in H^p$ and $g \in H^q$, where $p \geq 1$ and $1/p + 1/q = 1$. Let $K(z)$ be a function analytic in $|z| > r_0$, for some $r_0 < 1$, and $zf(z)(K(z) - g(z)) = R(z)$. If $R(e^{i\theta})$ is real, for almost all θ, then $R(z)$ extends analytically across ∂U.*

PROOF. It is clear that $R(z)$ is analytic in $r_0 < |z| < 1$. On $1 < |z| < 1/r_0$, define $R(z) = \overline{R(1/\bar{z})}$. Radial limits of $R(z)$ from both sides of ∂U are the same $(R(e^{i\theta}))$ almost everywhere. Let $C_{r_1} = \{z: |z| = r_1\}$ and $D_{r_1} = \{z: |z| = 1/r_1\}$, where $r_0 < r_1 < 1$. Consider

$$F(z) = \frac{1}{2\pi i} \int_{D_{r_1}} \frac{R(w)\,dw}{w - z} - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{R(w)\,dw}{w - z},$$

since $R(z)$ is analytic on C_{r_1} and D_{r_1}, $F(z)$ is analytic in $r_1 < |z| < 1/r_1$. For $1 > r > r_0/r_1$, the function $R(rz)$ is analytic in $r_1 \leq |z| \leq 1$. Hence

$$\frac{1}{2\pi i} \int_{\partial U} \frac{R(rw)\,dw}{w - z} - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{R(rw)\,dw}{w - z}$$

equals $R(rz)$ when $r_1 < |z| < 1$ and zero when $|z| > 1$. Then we conclude, by using (6), that

$$\frac{1}{2\pi i} \int_{\partial U} \frac{R(w)\,dw}{w - z} - \frac{1}{2\pi i} \int_{C_{r_1}} \frac{R(w)\,dw}{w - z}$$

equals $R(z)$ when $r_1 < |z| < 1$ and zero when $|z| > 1$. Similarly, one can conclude that

$$\frac{1}{2\pi i} \int_{D_{r_1}} \frac{R(w)\,dw}{w - z} - \frac{1}{2\pi i} \int_{\partial U} \frac{R(w)\,dw}{w - z}$$

equals $R(z)$ when $1 < |z| < 1/r_1$ and zero when $|z| < 1$. Therefore, $F(z) = R(z)$ for all z such that $r_1 < |z| < 1/r_1$ and $|z| \neq 1$. Thus $R(z)$ extends analytically across ∂U.

LEMMA 3. *Let $R(z)$ be a function analytic in $r_0 < |z| < t$, where $r_0 < 1$ and $t > 1$. If $R(z)$ does not vanish on ∂U then*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|R(e^{it})|\, dt \tag{14}$$

is analytic on $\bar{U}$.

PROOF. f is analytic in U. The fact that $R(z)$ is analytic and nonvanishing on ∂U implies that $\log|R(e^{i\theta})|$ is continuously differentiable and consequently $f(e^{i\theta})$ is continuous for all θ [4, p. 26].

Let $z_0 \in \partial U$ and let Δ be a small disc centered at z_0 so that $R(z)$ is analytic and does not vanish on $\bar{\Delta}$. Thus $\log R(z)$ has an analytic branch in Δ. (14) implies that $\operatorname{Re} f(z) = \operatorname{Re} \log R(z)$, for all $z \in \Delta \cap \partial U$. In other words, $i(f(z) - \log R(z))$ is real for all $z \in \Delta \cap \partial U$. This and the continuity of $f(z) - \log R(z)$, on $\Delta \cap \partial U$, give that $f(z) - \log R(z)$ can be continued analytically across $\Delta \cap \partial U$. Since $\log R(z)$ is analytic on $\Delta \cap \partial U$, $f(z)$ is analytic on $\Delta \cap \partial U$, and, in particular, at z_0.

LEMMA 4. *Let B_n be a finite Blaschke product, Q either a nonzero constant or a polynomial with all of its zeros on ∂U and h a nonvanishing analytic function on $\bar{U}$. Let*

$$K_1(e^{i\theta}) = \frac{1}{e^{i\theta} B_n(e^{i\theta})} \cdot \frac{|Q(e^{i\theta})|^2}{[Q(e^{i\theta})]^{2/p}} \cdot \frac{|h(e^{i\theta})|^p}{h(e^{i\theta})} \qquad (1 \leq p < \infty).$$

Then there are functions g and K such that $g \in H^\infty$, K is analytic in $|z| > r_0$, for some $r_0 < 1$, $K(\infty) = 0$ and $K_1(e^{i\theta}) = g(e^{i\theta}) + K(e^{i\theta})$.

PROOF. It is clear that $1/zB_n(z)$ is analytic in a neighborhood of ∂U. Write

$$Q(z) = c \prod_{j=1}^{m} (z - \alpha_j) \qquad (|\alpha_j| = 1).$$

Hence

$$|Q(e^{i\theta})|^2 = \frac{c_1}{e^{im\theta}} \prod_{j=1}^{m} \left(e^{i\theta} - \alpha_j\right)^2 \qquad (c_1 \text{ is constant}).$$

Let

$$l(z) = \frac{c_1}{z^m} \prod_{j=1}^{m} (z - \alpha_j)^2 \qquad (z \in U).$$

l is an analytic function in $\frac{1}{2} \leq |z| \leq 1$ and $l(e^{i\theta}) = |Q(e^{i\theta})|^2$. Since $Q(z) \neq 0$ for every $z \in U$, it follows that $[Q(z)]^{2/p}$ has a nonvanishing analytic branch. This, and the condition $p \geq 1$, imply that $l(z)/[Q(z)]^{2/p}$ is analytic and bounded in $\frac{1}{2} \leq |z| < 1$.

Let $S(z) = [h(\bar{z})]^{p/2}$. S is analytic on $\bar{U}$ because h is analytic and nonvanishing on $\bar{U}$. Also

$$S(e^{-i\theta})[h(e^{i\theta})]^{p/2} = |h(e^{i\theta})|^p.$$

Now, if we let

$$F(z) = \frac{1}{zB_n(z)} \cdot \frac{l(z)}{[Q(z)]^{2/p}} \cdot \frac{S(1/z)\cdot[h(z)]^{p/2}}{h(z)},$$

then F is a bounded analytic function in $r_0 < |z| < 1$, for some $r_0 < 1$, and $F(e^{i\theta}) = K_1(e^{i\theta})$. Furthermore, F has a Laurent expansion

$$F(z) = \sum_{j=0}^{\infty} a_j z^j + \sum_{j=0}^{\infty} \frac{b_j}{z^{j+1}} = g(z) + K(z)$$

where g is analytic in U and K is analytic in $|z| > r_0$. Since F is bounded and K is analytic on ∂U, it follows that $g \in H^\infty$.

We come now to the main result of the paper.

THEOREM 5. (a) *f is a support point of the unit ball of H^p, where $1 \leqslant p < \infty$, if and only if, $\|f\|_p = 1$ and f is of the form*

$$(15) \qquad f(z) = [Q(z)]^{2/p} \cdot W(z)$$

where Q is either a nonzero constant or a polynomial with all of its zeros on ∂U and W is a function analytic on $\overline{U}$ and nonvanishing on ∂U.

(b) *f is a support point of the unit ball of H^∞ if and only if, f is a finite Blaschke product.*

PROOF. (i) Suppose that f is a support point of the unit ball of H^p $(1 \leqslant p \leqslant \infty)$. There is a continuous linear functional L on A so that

$$\operatorname{Re} L(f) = \max\{\operatorname{Re} L(h) \colon h \in H^p, \|h\|_p \leqslant 1\}$$

and $\operatorname{Re} L$ is nonconstant. Since $e^{i\lambda} f \in H^p$ for any real λ, it follows that f also maximizes $|L|$. Assume that $L(f) > 0$. Let $K(z)$ be the function associated with L, as given by (1) and (2). Then there is a unique $g \in H^q$ $(1/p + 1/q = 1)$ so that f and g satisfy relations (9) through (13). Let

$$(16) \qquad R(z) = zf(z)(k(z) - g(z)).$$

$R(z)$ is analytic in some neighborhood of ∂U, by (10) and Lemma 2. Also, by (11), (12) and (16) we have, for $1 \leqslant p < \infty$, the relations

$$(17) \qquad |f(e^{i\theta})| = \frac{(R(e^{i\theta}))^{1/p}}{\|K - g\|_q^{1/p}} \quad \text{for almost all } \theta$$

and

$$(18) \qquad |K(e^{i\theta}) - g(e^{i\theta})| = \|K - g\|_q^{1/p}(R(e^{i\theta}))^{1/q} \quad \text{for almost all } \theta.$$

Hence we conclude, when $1 \leqslant p < \infty$, that f and g are bounded functions. When $p = \infty$, f is bounded and so the analyticity of $R(z)$ implies that $K(e^{i\theta}) - g(e^{i\theta}) \neq 0$ almost everywhere. Consequently, (13) implies that $|f(e^{i\theta})| = 1$ for almost all θ and then (16) implies that g is bounded. Therefore f and g are bounded for all $1 \leqslant p \leqslant \infty$.

If the zeros of f have an accumulation point on ∂U, then $R(z)$ would have zeros with an accumulation point on ∂U. This is impossible, because $R(z)$ is analytic in some neighborhood of ∂U. Thus, f has a finite number of zeros in U.

Let $S(z)$ be the singular inner factor of f. If $S(z)$ was not identically 1 then $S(z)$ would have either a zero of infinite order or an infinite number of zeros on ∂U [**3**, p. 76]. Since f and $K - g$ are bounded on ∂U, $R(z)$ would also have either a zero of infinite order or an infinite number of zeros on ∂U. But $R(z)$ is analytic on ∂U, so $S(z) \equiv 1$.

Hence f can be written

$$f(z) = B_n(z) f_1(z) \tag{19}$$

where $B_n(z)$ is a finite Blaschke product and $f_1(z)$ is an outer function. Since $R(z) \geqslant 0$ on ∂U, it follows that each zero of $R(z)$, on ∂U (if there is any) is an even order zero. Hence $R(z)$ can be written

$$R(z) = \prod_{k=1}^{m} (z - \alpha_k)^2 \cdot R_1(z) \tag{20}$$

where $|\alpha_k| = 1$, for $k = 1, 2, \ldots, m$ and $R_1(z)$ is analytic and does not vanish on ∂U. When $p = \infty$, $|f(e^{i\theta})| = 1$ almost everywhere. This and (19) imply that $f(z) = B_n(z)$ which is part (b). Assume for the rest of (i) that $1 \leqslant p < \infty$. We conclude, by (17) and (19), that

$$f_1(z) = C \exp\left\{ \frac{1}{2\pi p} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |R(e^{it})| \, dt \right\}$$

where C is a constant. Combine this with (20) to get

$$f_1(z) = C \prod_{k=1}^{m} (z - \alpha_k)^{2/p} \cdot \exp\left\{ \frac{1}{2\pi p} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |R_1(e^{it})| \, dt \right\}.$$

So, by Lemma 3,

$$f_1(z) = C \prod_{k=1}^{m} (z - \alpha_k)^{2/p} \cdot h(z)$$

where $h(z)$ is nonzero and analytic on $\overline{U}$. In specific, $f_1(z) = ch(z)$ in case $R(z)$ has no zero on ∂U. Write $W(z) = CB_n(z) \cdot h(z)$ to get statement (15).

(ii) Conversely, suppose that f has the form (15) and $\|f\|_p = 1$, where $1 \leqslant p < \infty$. Write $W = B_n h$, where B_n is a finite Blaschke product and h is a nonvanishing analytic function on $\overline{U}$. Let

$$K_1(e^{i\theta}) = \frac{|f(e^{i\theta})|^p}{e^{i\theta} f(e^{i\theta})}. \tag{21}$$

Then

$$K_1(e^{i\theta}) = \frac{1}{e^{i\theta} f(e^{i\theta})} \cdot \frac{|Q(e^{i\theta})|^2}{[Q(e^{i\theta})]^{2/p}} \cdot \frac{|h(e^{i\theta})|^p}{h(e^{i\theta})}$$

and consequently, by Lemma 4, $K_1(e^{i\theta}) = g(e^{i\theta}) + K(e^{i\theta})$, where g and K are as in the Lemma. Let L be the continuous linear functional on A given by K, as in (2). (8) and the fact that $g \in H^\infty$ imply that

$$L(G) = \frac{1}{2\pi i}\int_{|z|=1} G(z)K_1(z)\,dz \tag{22}$$

for every $G \in H^p$ $(p \geq 1)$. (21) implies that $|f(e^{i\theta})|^p = |K_1(e^{i\theta})|^q$ $(1/p + 1/q = 1)$, for $1 < p < \infty$, and $|K_1(e^{i\theta})| = 1$ for $p = 1$. Hence we conclude that

$$|L(G)| \leq \|G\|_p \leq 1 \qquad (1 \leq p < \infty)$$

for every G in the unit ball of H^p. Also, by (21) and (22), we have $L(f) = \|f\|_p^p = 1$. Therefore f is a support point of the unit ball of H^p.

When $p = \infty$, let f be a finite Blaschke product and

$$K_1(e^{i\theta}) = \frac{1}{e^{i\theta}f(e^{i\theta})}.$$

Apply Lemma 4 and then construct L as above to conclude that f is a support point of the unit ball of H^∞.

REMARKS. 1. Every point on the boundary of the unit ball of H^p $(1 < p < \infty)$ is an extreme point. Hence the set of support points of the unit ball of H^p $(1 < p < \infty)$ is much more restricted than the set of extreme points.

2. Extreme points of the unit ball of H^∞ are characterized by

$$\int_0^{2\pi} \log\left(1 - |f(e^{i\theta})|\right) d\theta = -\infty.$$

Hence the set of support points of the unit ball of H^∞ is much more restricted than the set of extreme points.

3. Extreme points of the unit ball of H^1 are characterized by $\|f\|_1 = 1$ and f is outer. So the set of extreme points is not a subset of the set of support points. $f(z) = z$ is a support point but not an extreme point. Thus, the set of support points is not, also, a subset of the set of extreme points.

I would like to thank the referee for suggesting the argument for the converse of Theorem 5.

ACKNOWLEDGMENT. I finally would like to thank the University of Petroleum and Minerals for their support and encouragement.

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