ON SPECTRAL SYNTHESIS FOR SETS OF THE FORM \( \text{int}(E) \)

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Abstract. The existence of a Helson set disobeying spectral synthesis is combined with the modified Herz criterion to construct a subset \( E \) of the circle such that spectral synthesis holds for \( E \) and fails for \( \partial E \).

In this note we study the spectral synthesis properties of sets \( E \) in the circle group \( T \) for which \( E = \text{int}(E) \). We use the notation of [2]. For \( E \subset T \), let \( \text{int}(E) \) denote the set of interior points of \( E \). A closed set \( E \) is a set of spectral synthesis, or an \( S \)-set, if, for any pseudomeasure \( S \) having support in \( E \), there is a net of measures \( \{\mu_\alpha\} \) supported by \( E \) so that \( \mu_\alpha \rightharpoonup S \). A set \( E \) is a Helson set if there exists a number \( B \), the Helson constant of \( E \), so that \( \|\mu\| \leq B\|\mu\|_{PM} \) for all \( \mu \in M(E) \). In the case that \( E \) is both a Helson set and an \( S \)-set, then every pseudomeasure supported by \( E \) is necessarily a measure (see [2, p. 92]). We prove the following result.

Theorem 1. There is a closed set \( E \subset T \) that satisfies spectral synthesis and yet spectral synthesis fails for the boundary set \( \partial E \).

The set \( E \) will have the form \( E = \text{int}(E) \) and will satisfy a modified Herz criterion: there exists \( 0 < \varepsilon < \frac{1}{2} \) and a sequence of positive integers \( \{p_k\} \) tending to infinity so that the sets

\[
H(E, p_k, \varepsilon) = \{x = 2\pi n/p_k : n \in \mathbb{Z} \text{ and dist}(x, E) < 2\pi(1 - \varepsilon)/p_k\}
\]

are all contained in \( E \). This ensures that for every \( S \in PM(E) \) there is a sequence \( \{\mu_k\} \) of measures supported by \( E \) satisfying \( \mu_k \rightharpoonup S \) and \( \|\mu_k\|_{PM} \leq B\|S\|_{PM} \), where the constant \( B \) depends only on the set \( E \) and not the particular pseudomeasure [2, p. 77]. This result is not new in that the unit ball \( E^n = \{x \in \mathbb{R}^n : |x| \leq 1\} \) satisfies a strong form of spectral synthesis, and it is well known that the boundary set, the unit sphere \( S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \), is a non-\( S \)-set for \( n > 2 \). Our result is new for the group \( T \).

We first give a lemma. We say that a set \( F \subset T \) is independent if it is independent in \( \mathbb{R} \) over the rationals, that is, given integers \( n_1, n_2, \ldots, n_m \) and distinct points \( x_1, x_2, \ldots, x_m \) in \( F \), then \( n_1x_1 + \cdots + n_mx_m = 0 \) implies \( n_1 = n_2 = \cdots = n_m = 0 \). Let \( F \subset T \) be an independent set that contains no rational multiples of \( \pi \), and let \( \{q_k\} \) be an increasing sequence of positive integers tending to infinity. Given an
element \( y \in T \), it is easy to find \( x_1 \) and \( x_2 \) disjoint from \( F \cup \{ r\pi: r \text{ rational} \} \) so that \( F \cup \{ x_1, x_2 \} \) is independent, \( y \) lies in the interval \( I = [x_1, x_2] \), and \( I \cap F = \emptyset \). Using a well-known method of constructing perfect, independent sets (see [3, pp. 101–102]), we can find \( x_1 \) and \( x_2 \) that satisfy the above conditions and the further condition that some subsequence \( \{ q_k^1 \}_{k=0}^{\infty} \) of \( \{ q_k \}_{k=1}^{\infty} \) exists for which the sets \( H(I, q_k^1, \frac{1}{4}), k \geq 0 \), are contained in \( I \). The same argument allows us to prove

**Lemma 2.** Let \( F_0 \) be an independent set in \( T \) that contains no rational multiples of \( \pi \), let \( \{ q_k \}_{k=0}^{\infty} \) be an increasing sequence of positive integers tending to infinity, and let \( H \) be a finite set with \( F_0 \cap H = \emptyset \). Then there exist disjoint intervals \( I_1, I_2, \ldots, I_m \) with \( I = \bigcup_{j=1}^{m} I_j \) and \( \partial I \cap \{ r\pi: r \text{ rational} \} = \emptyset \) satisfying

(i) \( I \cap F_0 = \emptyset \),
(ii) \( F_0 \cup \partial I \) is independent,
(iii) \( H \subseteq I \subseteq H + (-2\pi/4q_0, 2\pi/4q_0) \),
(iv) there exists a subsequence \( \{ q_k^1 \}_{k=0}^{\infty} \) of \( \{ q_k \}_{k=1}^{\infty} \) so that the sets \( H(I, q_k^1, \frac{1}{4}), k \geq 0 \), are contained in \( I \).

**Proof of the Theorem.** Let \( F \) be an independent Helson set in \( T \) for which spectral synthesis fails [2, p. 118]. Since \( F \) is independent, we can assume that \( F \) contains no rational multiples of \( \pi \). We use the well-known fact that if \( F \) is a given Helson set, then for any finite independent set \( H \), \( F \cup H \) is a Helson set whose Helson-set constant is bounded by a fixed constant \( B \) depending only on the Helson-set constant of the set \( F \) (see [2, p. 51]). We define inductively sets \( I_n, n \geq 1 \), each of which is a finite union of closed intervals, and a sequence \( \{ p_k \}_{k=1}^{\infty} \) of positive integers. Let \( p_1 = q_0 = 2 \), and use Lemma 2 to obtain \( I_1 \) and a subsequence \( \{ q_k^1 \}_{k=0}^{\infty} \) of positive integers. After having chosen sets \( I_1, \ldots, I_{n-1} \) and integers \( p_1, \ldots, p_{n-1} \) and obtaining a subsequence \( \{ q_k^{n-1} \}_{k=0}^{\infty} \), choose \( p_n \in \{ q_k^{n-1} \} \) large enough so that

\[
2\pi/p_n < 10 \min \left\{ \text{dist}(x, F): x \in \bigcup_{j=1}^{n-1} I_j \right\}.
\]  

Let \( H_n \) denote the set \( H(F, p_n, 0) \) and apply the lemma with \( q_0 = p_n \) to obtain a finite collection of intervals whose union \( I_n \) satisfies the conclusions of Lemma 2 with \( F_0 = F \cup \bigcup_{k=1}^{n-1} \partial I_k \), \( H = H_n \), and some subsequence \( \{ q_k^n \}_{k=0}^{\infty} \) of \( \{ q_k^{n-1} \}_{k=0}^{\infty} \).

Now define \( E = F \cup \bigcup_{n=1}^{\infty} I_n \). It is clear that \( E = \bigcup I_n = \text{int}(E) \). We claim that \( E \) satisfies a modified Herz criterion for the sets (1) for the sequence \( \{ p_n \} \) and for \( \varepsilon = \frac{1}{4} \). Let \( \varepsilon = \frac{1}{4}, k \in \mathbb{Z} \) and \( x = 2\pi k/p_n \) satisfy \( \text{dist}(x, E) < 2\pi(1-\varepsilon)/p_n \). If \( x \in H_n \subseteq I_n \subseteq E \), there is nothing to prove, so assume \( x \notin H_n \). Then \( \text{dist}(x, F) > 2\pi/p_n \), and \( \text{dist}(x, I_m) < 2\pi(1-\varepsilon)/p_n \) for some \( m \). Since (2) implies that \( I_m \subseteq F + (-2\pi/p_n, 2\pi/p_n) \) for \( m > n \), and since \( m = n \) implies \( x \in H_n \), we in fact have \( m < n \). Property (iv) and the fact that \( p_n \in \{ q_k^{m-1} \} \) for \( m < n \) now forces \( x \in I_m \subseteq E \). Thus, \( E \) is a set of synthesis.

To finish the proof we show that \( \partial E = F \cup \bigcup_{n=1}^{\infty} \partial I_n \) is a Helson non-\( S \)-set. Let \( \mu \in M(\partial E) \) and \( \varepsilon > 0 \) be given. Since \( \bigcup_{n=1}^{\infty} \partial I_n \) is a countable set, we can find an integer \( N \) so that the measure \( \mu_N \), the restriction of \( \mu \) to the set \( F \cup \bigcup_{n=1}^{N} \partial I_n \), has
By construction, \( \bigcup_{n=1}^{N} \partial I_n \) is a finite independent set, and so we obtain
\[
\|\mu\| < \|\mu_N\| + \varepsilon < B\|\mu_N\|_{PM} + B\varepsilon + \varepsilon.
\]
Since \( \varepsilon \) and \( \mu \) are arbitrary, \( \|\mu\| < B\|\mu\|_{PM} \) for all measures \( \mu \) supported by \( \partial E \), i.e., \( \partial E \) is a Helson set. Since there exists an \( S \in PM(F) \subseteq PM(\partial E) \) that is not a measure, this proves that \( \partial E \) is a non-\( S' \)-set.

**Remarks.** 1. The proof of the theorem is easily adapted so the set \( E \) satisfies a modified Herz criterion with sets \( H(E, p_k, \varepsilon) \) in (1) for any \( \varepsilon \) with \( 0 < \varepsilon < \frac{1}{2} \).

2. A similar proof yields the existence of sets \( E \) which are the closures of their interiors and for which spectral synthesis fails. For let \( F \) be a non-\( S \)-set in \( T \). Then there exists a \( \phi \in A(T) \) and an \( S \in PM(F) \) satisfying \( \phi = 0 \) on \( F \) and \( \langle S, \phi \rangle \neq 0 \) \( [2, \text{p. 69}] \). If \( \{x_n\}_{n=1}^{\infty} \subseteq F \) is dense in \( F \), choose \( y_n \notin F, n \geq 1 \), with \( \text{dist}(x_n, y_n) \to 0 \) and \( |\phi(y_n)| < 2^{-2n} \). We can now find functions \( \phi_n \in A(T) \) with mutually disjoint supports and supports disjoint from \( F \) so that \( \|\phi_n\| < 2^{-n} \) and \( \phi_n = \phi \) on some interval \( I_n \) containing \( y_n \). Since the function \( \phi - \Sigma \phi_n \) belongs to \( A(T) \), vanishes on \( E = F \cup \bigcup_{n=1}^{\infty} I_n \), and has \( \langle S, \phi - \Sigma \phi_n \rangle = \langle S, \phi \rangle \neq 0 \), the set \( E = \text{int}(E) \) disobeys synthesis. The existence of non-\( S \)-sets which are closures of their interiors was originally suggested by Beurling \([1]\).