THE UNIQUENESS OF MULTIPLICATION
IN FUNCTION ALGEBRAS

KRZYSZTOF JAROSZ

Abstract. Let $A$ be a function algebra. We prove that the original multiplication of $A$ is a unique multiplication on the underlying Banach space of $A$ which produces a Banach algebra with the same unit as the original one.

Let $A$ be a complex Banach algebra with unit. We denote by $1$ the unit of $A$, the norm by $\| \cdot \|$ and the product of $f$ and $g$ by $fg$ or $f \cdot g$. By the definition of a Banach algebra, for any elements $f$, $g$ in $A$ we have

\[(1)\quad \| f \cdot g \| \leq \| f \| \| g \| \]
and

\[(2)\quad 1 \cdot f = f.\]

Suppose now that $A$ is uniform algebra, that is, $A$ is a commutative Banach algebra with unit and $\| f^2 \| = \| f \|^2$ for all $f$ in $A$. Our goal is to prove that there exists exactly one (associative) multiplication on the Banach space $A$ which satisfies (1) and (2). This result follows upon considering a more general situation.

By an $\varepsilon$-deformation of $A$ we mean an associative multiplication $\times$ on the Banach space $A$ such that

\[(3)\quad \| f \times g - f \cdot g \| \leq \varepsilon \| f \| \| g \| \quad \text{for all } f, g \in A.\]

This definition was formulated by Johnson [2] (see also [3,4]). He investigates whether all multiplications on a Banach algebra $A$ near the given multiplication share some of the properties of the original one. Small deformations of function algebras were studied deeply by R. Rochberg [5].

If $\times$ is an $\varepsilon$-deformation of the multiplication of a Banach algebra $A$ then for all $f$, $g$ in $A$

\[(4)\quad \| f \times g \| \leq (1 + \varepsilon) \| f \| \| g \| \]
and

\[(5)\quad \| 1 \times f - f \| \leq \varepsilon \| f \|.\]

Our main theorem shows that for uniform algebras the converse implication also holds.

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Theorem 1. Suppose \((A, \cdot)\) is a complex uniform algebra. There are positive constants \(\varepsilon_0, c\) which do not depend on \(A\) such that for any \(0 < \varepsilon \leq \varepsilon_0\) and any multiplication with unit on \(A\) satisfying the conditions

(i) \[\|f \times g\| \leq (1 + \varepsilon)\|f\|\|g\|\]

(ii) \[\|1 \times f - f\| \leq \varepsilon\|f\|\text{ for all } f, g \in A,\]

(iii) \[\|f \times g - f \cdot g\| \leq c\varepsilon\|f\|\|g\|\text{ for all } f, g \in A.\]

Moreover the new multiplication \(\times\) is commutative.

Proof. If \(\varepsilon_0 < 1\) then the condition (ii) implies that the operator \(T: A \to A: f \mapsto 1 \times f\) is an isomorphism so there exists an element \(e\) of \(A\) such that \(1 \times e = 1\). It is easy to check that \(e\) is the unit of the algebra \((A, \times)\) and that the element \(1\) is invertible in this algebra. A simple computation using (i) and (ii) proves that

\[\|f \times g - 1^{-1} \times f \times g\| \leq \frac{2(1 + \varepsilon)^2\varepsilon}{1 - \varepsilon}\|f\|\|g\|\]

for all \(f, g\) in \(A\). Hence the multiplication \(\hat{\times}\) defined by \(\hat{f} \hat{\times} g = 1^{-1} \times f \times g\) has the same unit as the original multiplication of the function algebra \(A\) and the multiplication denoted by \(\hat{\times}\) is a \(ke\)-deformation of the multiplication \(\times\). This proves that without loss of generality we may assume that the element \(1\) is a common unit of both multiplications \(\times\) and \(\hat{\times}\).

Let us now introduce some notation.

By \(\partial A\) and \(Ch A\) we denote the Shilov and the Choquet boundaries of \(A\), respectively. Let

\[\Omega = \left\{ x + iy \in \mathbb{C}: (x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 < \frac{1}{2} \right\} \cap \left\{ x + iy \in \mathbb{C}: (x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 < \frac{1}{2} \right\}\]

and for \(r > 0\)

\[D(r) = \{ x + iy \in \mathbb{C}: x^2 + y^2 < r^2 \} \]

Notice that without loss of generality we may assume that \(A\) is an algebra of continuous functions on \(\partial A\).

Fix \(\delta > 0\), and let \(\kappa: \overline{D(1)} \to \overline{\Omega}\) be a continuous map of \(\overline{D(1)}\) onto \(\overline{\Omega}\) such that \(\kappa\) is analytic on \(\overline{D(1)}\) and

\[\kappa(1) = 1 \quad \text{and} \quad \kappa(0) = \delta/2\]

Let \(V \subset \mathbb{C}\) be a neighborhood of \(0\) such that

\[\kappa(V) \subset \overline{\Omega} \cap D(\delta)\]

Now fix any point \(s_0 \in Ch A\) and any of its neighborhoods \(U \subset \partial A\), and let \(f \in A\) be such that

\[\|f\| = f(s_0) = 1 \quad \text{and} \quad f(\partial A - U) \subset V\]

The function \(\kappa \circ f \in A\) has the following properties:

(a) \(\kappa \circ f(\partial A) \subset \overline{\Omega}\);
(b) \(\|\kappa \circ f\| = \kappa \circ f(s_0) = 1\);
(c) \(\kappa \circ f(\partial A - U) \subset \overline{\Omega} \cap D(\delta)\).
Hence for any \( s_0 \in \text{Ch} \ A \) there exists a net \((f_\alpha) \subset A\) such that
(A) \( f_\alpha(\partial A) \subset \bar{\Omega} \),
(B) \( \|f_\alpha\| = f_\alpha(s_0) = 1 \),
(C) \((f_\alpha)\) tends uniformly to zero on the compact subsets of the set \( \partial A - \{s_0\} \).

Using the net \((f_\alpha)\) we define
\[
g'_\alpha = f_\alpha + i(1 - f_\alpha), \quad g''_\alpha = f_\alpha - i(1 - f_\alpha).
\]
By direct computation
\[
g'_\alpha \times g'_\beta = f_\alpha + f_\beta - 1 + i(f_\alpha + f_\beta - 2f_\alpha \times f_\beta).
\]
Further observe that, by the definition of \( \Omega \), we have
\[
\|g'_\alpha\| = \sup_{s \in \partial A} |f_\alpha(s) + i(1 - f_\alpha(s))| \leq \sup_{z \in \Omega} |z + i(1 - z)| = 1.
\]

Hence from (i) we get
\[
1 + \epsilon \geq \|g'_\alpha \times g'_\beta\| \geq |g'_\alpha \times g'_\beta(s_0)| = |1 + 2i(1 - f_\alpha \times f_\beta(s_0))|.
\]

The same computations for the functions \( g''_\alpha \) and \( g''_\beta \) show that
\[
1 + \epsilon \geq \|g''_\alpha \times g''_\beta\| \geq |g''_\alpha \times g''_\beta(s_0)| = |1 - 2i(1 - f_\alpha \times f_\beta(s_0))|.
\]

Inequalities (6) and (7) can be satisfied simultaneously only if
\[
|1 - f_\alpha \times f_\beta(s_0)| \leq \sqrt{\epsilon/2 + \epsilon^2/4} \leq \sqrt{\epsilon}.
\]

Now for any \( g \in A \) define two functionals \( T'_g: A \rightarrow \mathbb{C} \) and \( T'_g: A \rightarrow \mathbb{C} \) by
\[
T'_g(f) = g \times f(s_0), \quad T'_g(f) = f \times g(s_0).
\]
For each \( g \in A \) fix two regular measures \( \mu'_g \) and \( \mu'_g \) on \( \partial A \) such that
\[
\mu'_g(f) = T'_g(f), \quad \var(\mu'_g) = \|T'_g\|,
\]
\[
\mu'_g(f) = T'_g(f), \quad \var(\mu'_g) = \|T'_g\| \quad \text{for all } f \in A.
\]
Inequality (8) shows that
\[
|\mu'_\alpha(f_\beta) - 1| \leq \sqrt{\epsilon} \quad \text{for any } \alpha \text{ and all } \beta.
\]
By the definition of \((f_\alpha)\) we get
\[
|\mu'_\alpha(\{s_0\}) - 1| \leq \sqrt{\epsilon}.
\]
Hence, because \( \var(\mu'_\alpha) = \|T'_\alpha\| = 1 + \epsilon \), the measure \( \mu'_\alpha \) is of the form
\[
\mu'_\alpha = \delta_{s_0} + \Delta \mu'_\alpha,
\]
where \( \delta_{s_0} \) is a Dirac measure concentrated at the point \( s_0 \) and \( \var(\Delta \mu'_\alpha) \leq 3\sqrt{\epsilon} \).

Now let \( g_0 \) be any element of \( A \) such that \( \|g_0\| = 1 = g_0(s_0) \). By (10) we get
\[
\mu'_{g_0}(f_\alpha) = f_\alpha \times g_0(s_0) = \mu'_\alpha(g_0)
\]
\[
= g_0(s_0) + \Delta \mu'_\alpha(g_0) = 1 + \Delta \mu'_\alpha(g_0).
\]
Hence
\[
|\mu'_{g_0}(f_\alpha) - 1| \leq 3\sqrt{\epsilon}.
\]
In the same way as previously, we get
\[(11) \quad \mu'_{g_0} = \delta_{s_0} + \Delta \mu'_{g_0}, \quad \text{where} \quad \text{var}(\Delta \mu'_{g_0}) \leq \frac{7}{\sqrt[4]{\epsilon}}.\]

Using this we can estimate the norm of $g_0 \times g_0$ from below.
\[
\|g_0 \times g_0\| \geq |g_0 \times g_0(s_0)| = |\mu'_{g_0}(g_0)| = |1 + \Delta \mu'_{g_0}(g_0)| \\
\geq 1 - \frac{7}{\sqrt[4]{\epsilon}}.
\]

Because $s_0$ is an arbitrary point of $\text{Ch} \ A$ this proves that
\[(12) \quad \|g \times g\| \geq \left(1 - \frac{7}{\sqrt[4]{\epsilon}}\right)\|g\|^2 \quad \text{for any} \quad g \in A.
\]

As an immediate consequence of (12) we conclude that the spectral radius of any element $g$ of the algebra $(A, \times)$ is not less than $(1 - \frac{7}{\sqrt[4]{\epsilon}})\|g\|$. Hence by a theorem of Hirschfeld and Żelazko [1] one obtains the commutativity of the multiplication $\times$ if $1 - \frac{7}{\sqrt[4]{\epsilon}} > 0$.

Applying (12) for $g = f_\alpha$ and using the commutativity of $\times$ we get that there exists a linear and $\times$-multiplicative functional $F_\alpha$ such that $|F_\alpha(f_\alpha)| \geq 1 - \frac{7}{\sqrt[4]{\epsilon}}$. For any $f$ in $A$ of norm equal one we have
\[
(1 + \epsilon)\|F_\alpha\| \geq \|F_\alpha\|\|f \times f\| \geq |F_\alpha(f \times f)| = |F_\alpha(f)|^2,
\]

hence
\[
(1 + \epsilon)\|F_\alpha\| \geq \|F_\alpha\|^2, \quad \text{so} \quad \|F_\alpha\| \leq 1 + \epsilon.
\]

Let $\nu_\alpha$ be a regular measure on $\partial A$ which represents the functional $F_\alpha$ and such that $\text{var}(\nu_\alpha) = \|F_\alpha\|$. We have
\[
\begin{aligned}
|\nu_\alpha(f_\alpha)| &\geq 1 - \frac{7}{\sqrt[4]{\epsilon}}, \\
\text{var}(\nu_\alpha) &\leq 1 + \epsilon, \\
\nu_\alpha(1) &\leq 1 \quad \text{for all indices} \quad \alpha.
\end{aligned}
\]

Taking a net finer than $(f_\alpha)$ and using the weak $\ast$ compactness of $\partial A$ we can assume, without loss of generality, that the net $(\nu_\alpha)$ is weak $\ast$ convergent to the measure $\nu_0$. The measure $\nu_0$ also represents a linear and $\times$-multiplicative functional $F_0$ on $A$.

From (13) we derive that the measure $\nu_0$ is of the form
\[(14) \quad \nu_0 = \delta_{s_0} + \Delta \nu_{s_0} \quad \text{where} \quad \text{var}(\Delta \nu_{s_0}) \leq c_1\sqrt[4]{\epsilon}.
\]

From (14) for any $f$, $g$ in $A$ we find
\[
f \times g(s_0) + \Delta \nu_{s_0}(f \times g) = \nu_0(f \times g) = \nu_0(f) \cdot \nu_0(g) \\
= (f(s_0) + \Delta \nu_{s_0}(f)) \cdot (g(s_0) + \Delta \nu_{s_0}(g)) \\
= f(s_0) \cdot g(s_0) + \Delta \nu_{s_0}(f) \cdot g(s_0) + f(s_0) \cdot \Delta \nu_{s_0}(g) + \Delta \nu_{s_0}(f) \cdot \Delta \nu_{s_0}(g).
\]

Hence
\[
|f \times g(s_0) - f \cdot g(s_0)| \leq \text{var}(\Delta \nu_{s_0})(2\|f\|\|g\| + \text{var}(\Delta \nu_{s_0}) \cdot \|f\|\|g\|) \\
\leq c_1\sqrt[4]{\epsilon} \left(2 + c_1\sqrt[4]{\epsilon}\right) \cdot \|f\|\|g\| = c_1\sqrt[4]{\epsilon} \|f\|\|g\|.
\]
Because $s_0$ is an arbitrary point in $\text{Ch} A$, which is a dense subset of $\partial A$, the above statement proves (iii) and ends the proof of the theorem.

**Corollary 1.** Suppose $(A, \cdot)$ is a complex function algebra. Let $\times$ be any associative multiplication on the Banach space $A$ with the same unit and such that $(A, \times)$ is a Banach algebra (this means such that $\|f \times g\| \leq \|f\|\|g\|$ for all $f, g$ in $A$). Then the new multiplication $\times$ and the original one coincide.

The above corollary can be also formulated in the following way, giving a generalization of Nagasawa's Theorem.

**Corollary 2.** Let $(A, \cdot)$ be a complex function algebra with unit $1_A$ and let $B$ be any Banach algebra with unit $1_B$. Suppose $T$ is a linear isometry from $A$ onto $B$ such that $T1_A = 1_B$. Then $T$ is an algebra isomorphism of $A$ and $B$.

Notice that we have only considered complex Banach algebras. The theorem and the corollaries are not valid for real function algebras, even in two dimensions. To prove this let $A = (\mathbb{R}^2, \cdot, \|\cdot\|_{\infty})$ be the two dimensional real function algebra and let $\rho_t: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$.

$$\rho_t((x, y), (x', y')) = (xx' - t(x - y)(x' - y'), yy' - t(x - y)(x' - y')).$$

A direct computation shows that for any $0 \leq t \leq 1/2$ the bilinear map $\rho_t$ is a commutative, associative multiplication on $\mathbb{R}^2$ such that $\|\rho_t\| = 1$ and

$$\rho_t((1, 1), (x, y)) = (x, y) \quad \text{for any} \quad (x, y) \in \mathbb{R}^2.$$

Let us end the paper with a natural problem arising from Corollary 1.

**Problem.** Characterize those Banach spaces $A$ and elements $e$ which admit a unique multiplication $\times$ on $A$ so that $(A, \times)$ is a Banach algebra with unit $e$.

**References**


Institute of Mathematics, Warsaw University, Warsaw, Poland