SETS OF UNIQUENESS FOR A CERTAIN CLASS \( \mathbb{M}_e \)
ON THE DYADIC GROUP

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Abstract. For each sequence \( e = (e_n) \) of real numbers which satisfies
\[ \liminf_{n \to \infty} e_{2^n+1}/e_{2^n} > 0 \] and \( e_n \downarrow 0 \) as \( n \to \infty \), let \( \mathbb{M}_e \) denote the set of all Walsh series \( \mu \sim \sum_{k=0}^\infty \hat{\mu}(k)w_k(x) \) such that \( \sum_{k=0}^\infty |\hat{\mu}(k)|^2 < \infty \). We give a necessary and sufficient condition for a subset of the dyadic group to be a set of uniqueness for \( \mathbb{M}_e \).

1. Introduction. Wade [3] discussed whether or not a subset of the dyadic group is a set of uniqueness for \( \mathbb{M}_e \) and gave the complete solution for this problem. In this paper we shall partially generalize his result.

First we introduce some definitions and notations. Let \( I_0 \) be the dyadic group introduced by Fine [1], i.e., a compact abelian group with the additive operation \( + \). A dyadic interval of rank \( n \), \( I_n \), is the set of all 0-1 series, \( x = (t_1, t_2, \ldots) \), such that \( \sum_{k=1}^n t_k/2^k = p/2^n \). The dyadic interval of rank \( n \) which contains \( x \) is denoted by \( I_n(x) \). For convenience, when \( \lim_{k \to \infty} t_k \neq 1 \), we shall identify \( x = (t_1, t_2, \ldots) \) with \( (2^k t_k/2^k) \), and when \( \lim_{k \to \infty} t_k = 1 \), we shall write \( x = (\sum_{k=1}^\infty t_k/2^k) \).

Let
\[ \mu \sim \sum_{k=0}^\infty \hat{\mu}(k)w_k(x) \]
be an arbitrary Walsh series. Set
\[ m_\mu(I) = \lim_{N \to \infty} \int \sum_{k=0}^N \hat{\mu}(k)w_k(x) \, dx \]
for each dyadic interval \( I \). The limit of the right-hand side exists and is finite. Moreover, we have
\[ m_\mu(I_n(x)) = \frac{1}{2^n} \sum_{k=0}^{2^n-1} \hat{\mu}(k)w_k(x). \]
The set function \( m_\mu \) satisfies the following additive law:
\[ m_\mu(I_n^p) = m_\mu(I_n^{2p+1}) + m_\mu(I_n^{2p+1}) \]
for \( n = 0, 1, 2, \ldots, p = 0, 1, \ldots, 2^n - 1 \). By an easy calculation we can prove that
\[ \hat{\mu}(k) = \int_{I_0} w_k(x) m_\mu(dx), \]

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where \( \int m_\mu (dx) = m_\mu (I) \). Conversely, when a set function \( m \) satisfies (1), if we set

\[
\hat{\gamma}(k) = \int f_0 w_k(x) m(dx)
\]

for all \( k \), then

\[
\gamma \sim \sum_{k=0}^{\infty} \hat{\gamma}(k) w_k(x)
\]

is the Walsh Fourier series of \( m = m_\gamma \). We call a set function satisfying (1) a dyadic measure. For details of dyadic measures and the dyadic group, see [1 and 4].

When \( \varepsilon = \{ \varepsilon_k \}_k \) is a sequence of positive numbers such that

\[
\begin{align*}
\varepsilon_0 &= 1, \\
\varepsilon_k &= 0 \text{ as } k \to \infty, \\
\liminf_{n \to \infty} \varepsilon_{2^n+1}/\varepsilon_{2^n} &\equiv \theta > 0,
\end{align*}
\]

let \( \mathcal{M}_\varepsilon \) denote the set of all Walsh series \( \mu \) which satisfy \( \sum_{k=0}^{\infty} \varepsilon_k |\hat{\mu}(k)|^2 < \infty \). A subset \( E \) of the dyadic group is said to be a set of uniqueness for \( \mathcal{M}_\varepsilon \) if \( \mu \in \mathcal{M}_\varepsilon \) and

\[
\lim_{n \to \infty} \sum_{k=0}^{2^n-1} \hat{\mu}(k) w_k(x) = 0, \quad \text{except perhaps on } E,
\]

imply \( \mu \) is the zero series. If a subset is not a set of uniqueness for \( \mathcal{M}_\varepsilon \), it is called a set of multiplicity for \( \mathcal{M}_\varepsilon \). Notice when \( \varepsilon_0 = 1, \varepsilon_k = 1/k^{1-a}, k = 1,2,\ldots, \) where \( 0 < a < 1 \), that \( \mathcal{M}_\varepsilon \) coincides with \( \mathcal{M}_\alpha \) (see [3]). Since \( \mathcal{T}^+_\alpha \subset \mathcal{T}_\alpha \), if a subset of measure zero is a set of multiplicity for \( \mathcal{T}^+_\alpha \), then it is also a set of multiplicity for \( \mathcal{T}_\alpha \). Since the characteristic function \( f \) of a set of positive measure is an \( L^2 \)-integrable function, it satisfies

\[
\sum_{k=0}^{\infty} \varepsilon_k |\hat{f}(k)|^2 \leq \sum_{k=0}^{\infty} |\hat{f}(k)|^2 < \infty.
\]

Hence any set of positive measure is a set of multiplicity for each \( \mathcal{M}_\varepsilon \).

We shall prove the following theorems.

**Theorem 1.** Let \( \{n_k\}_k \) be a sequence of integers and set \( N_k = n_1 + n_2 + \cdots + n_k, N_0 = 0 \). Suppose \( E \) is a subset of \( I_0^p \) which satisfies the following property.

\[
E \cap I_{N_{k+1}}^{2p+s(p,k)} = \emptyset \quad \text{for } k = 1,2,\ldots, \quad p = 0,1,2,\ldots,2N_k - 1
\]

and \( s(p,k) = 0 \) or 1.

Then \( E \) is a set of uniqueness for \( \mathcal{M}_\varepsilon \) if and only if \( \sum_{k=1}^{\infty} \varepsilon_{2^{N_k}} \cdot 2^k = \infty \).

**Theorem 2.** It is sufficient for a closed set of measure zero to be a set of uniqueness for \( \mathcal{M}_\varepsilon \) if it satisfies

\[
\sum_{n=0}^{\infty} \frac{\varepsilon_{2^n} - \varepsilon_{2^{n+1}}}{\text{meas } E_{n+1}} = \infty
\]

or

\[
\limsup_{n \to \infty} \frac{\varepsilon_{2^n}}{\text{meas } E_n} = \infty,
\]

where \( E_n = \bigcup_{x \in E} I_{n}^p (x) \).
2. Proof of Theorem 1. Let $m_\mu$ be the dyadic measure associated with $\mu$. Since
\[
\sum_{k=2^{n}}^{2^{n+1}-1} |\hat{\mu}(k)|^2 = 2^n \sum_{p=0}^{2^n-1} |\Delta m_\mu(I^p_n)|^2,
\]
where
\[
\Delta m_\mu(I^p_n) = m_\mu(I^{2p+1}_n) - m_\mu(I^{2p+1}_n),
\]
we have
\[
\sum_{k=2^{n}}^{2^{n+1}-1} \varepsilon_k |\hat{\mu}(k)|^2 \leq 2^n \sum_{k=2^{n}}^{2^{n+1}-1} |\hat{\mu}(k)|^2 = \varepsilon_{2^n} \cdot 2^n \cdot \sum_{p=0}^{2^n-1} |\Delta m_\mu(I^p_n)|^2
\]
and
\[
\sum_{k=2^{n}}^{2^{n+1}-1} \varepsilon_k |\hat{\mu}(k)|^2 \geq \varepsilon_{2^{n+1}} \sum_{k=2^{n}}^{2^{n+1}-1} |\hat{\mu}(k)|^2 \geq \varepsilon_{2^n} \cdot 2^n \cdot \sum_{p=0}^{2^n-1} |\Delta m_\mu(I^p_n)|^2.
\]
Therefore, it is easy to see that
\[
\sum_{k=0}^\infty \varepsilon_k |\hat{\mu}(k)|^2 = \infty
\]
if and only if
\[
\sum_{n=0}^\infty \varepsilon_{2^n} \cdot 2^n \sum_{p=0}^{2^n-1} |\Delta m_\mu(I^p_n)|^2 = \infty.
\]
Let $m$ be the positive dyadic measure constructed by the following rule when $0 \leq n \leq n_1 = N_1$. Set $m(I^p_n) = 1/2^n$ and
\[
m(I^p_{N_{k+1}}) = \begin{cases} 1/2^{N_k} & \text{if } I^p_{N_k+1} \cap E \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}
\]
Continuing in this way, when $N_k < n \leq N_{k+1}$, $k = 1, 2, \ldots$, set
\[
m(I^p_n) = \begin{cases} 1/2^{n-k} & \text{if } I^p_n \cap E \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}
\]
From the definitions of $m$ and $\Delta m$, we have
\[
|\Delta m(I^p_n)| = \begin{cases} 1/2^{N_k-k+1} & \text{if } I^p_{N_k} \cap E \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}
\]
and when $n \neq N_k$, $\Delta m(I^p_n) = 0$ for all $p$. Set $m_{\mu_E} = m$. Since $\mu_E(0) = 1$, the Walsh series $\mu_E$ is not the zero series, and the dyadic measure $m_{\mu_E}$ satisfies the following equality:
\[
\sum_{n=0}^\infty \varepsilon_{2^n} \cdot 2^n \sum_{p=0}^{2^n-1} |\Delta m_{\mu_E}(I^p_n)|^2 = \sum_{k=1}^{\infty} 2^{N_k} \cdot \varepsilon_{2^{N_k}} \cdot 2^{N_k-k+1} (1/2^{N_k-k+1})^2
\]
\[
= \frac{1}{2} \sum_{k=1}^{\infty} \varepsilon_{2^{N_k}} \cdot 2^k.
\]
Therefore, if the last formula is finite, $E$ is a set of multiplicity for $\mathcal{M}_\mu$. 
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On the other hand, let $m'$ be a dyadic measure satisfying the following properties:

(2)  \( m'(I_0^0) = 1; \quad (ii) m'(I) = 0 \) if \( I \cap E = \emptyset \),

where \( I \) is an arbitrary dyadic interval. Therefore we have

\[
1 = |m'(I_0^0)| = \left| \sum_{p=0}^{2^{N_k} - 1} m'(I_{k_p}^0) \right| \\
= \left| \sum_{(I_{k_p}^0 \cap E = \emptyset)} m'(I_{k_p}^0) \right| \leq \sum_{(I_{k_p}^0 \cap E = \emptyset)} |\Delta m'(I_{k_p}^0)| \\
\leq \left\{ \sum_{(I_{k_p}^0 \cap E = \emptyset)} |\Delta m'(I_{k_p}^0)| \right\}^{1/2} \cdot \left\{ \sum_{(I_{k_p}^0 \cap E = \emptyset)} 1 \right\}^{1/2} \\
= \left\{ \sum_{(I_{k_p}^0 \cap E = \emptyset)} |\Delta m'(I_{k_p}^0)| \right\}^{1/2} \cdot \{2^{(N_k - k)/2}\}.
\]

From the above just cited, we have

\[
2^{k - N_k} \leq \sum_{p=0}^{2^{N_k} - 1} |\Delta m'(I_{k_p}^0)|^2.
\]

Let \( \mu \) be the Walsh Fourier series of \( m' \). Then

\[
\sum_{k=1}^{\infty} \epsilon_{2^{N_k}} 2^k \leq \sum_{k=1}^{2^{N_k} - 1} 2^{N_k} \sum_{p=0}^{2^{N_k} - 1} |\Delta m'(I_{k_p}^0)|^2 \\
\leq 2 \cdot \sum_{k=1}^{\infty} \epsilon_{2^{N_k}} \cdot \sum_{n=2^{N_k} - 1}^{2^{N_k}} |\hat{\mu}(n)|^2 \\
\leq 2 \cdot \frac{1}{\theta} \cdot \sum_{k=0}^{\infty} \epsilon_k \cdot |\mu(k)|^2.
\]

If \( E \) satisfies the hypothesis, then \( m' \) does not belong to \( \mathcal{M}_\varepsilon \). Namely, \( E \) is a set of uniqueness for \( \mathcal{M}_\varepsilon \). The proof is complete.

If the sequence \( \{N_k\}_k \) satisfies \( \lim_{k \to \infty} (N_k - k) / N_k = \alpha \), then the Hausdorff dimension of \( E \) is \( \alpha \) and, for each \( \varepsilon > 0 \), \( E \) satisfies

\[
\lim_{k \to \infty} 2^{N_k - k + 1} \cdot (1/2^{N_k})^\alpha \varepsilon = 0, \quad \lim_{k \to \infty} 2^{N_k - k + 1} \cdot (1/2^{N_k})^{\alpha - \varepsilon} = \infty,
\]

that is, the \( (\alpha + \varepsilon) \)-Hausdorff measure of \( E \) is zero and the \( (\alpha - \varepsilon) \)-Hausdorff measure of \( E \) is positive. Consequently we know that \( E \) is of \( (\alpha + \varepsilon) \)-capacity zero and of \( (\alpha - \varepsilon) \)-capacity positive (see [2]). A corollary of [3], then, is that \( E \) is a set of uniqueness for \( \mathcal{M}_{(\alpha + \varepsilon)} \) and a set of multiplicity for \( \mathcal{M}_{(\alpha - \varepsilon)} \). Theorem 1 improves this corollary because such a set \( E \) satisfies

\[
\sum_{k=1}^{\infty} 1/2^{N_k - (1 - (\alpha + \varepsilon))} \cdot 2^k = \infty, \quad \sum_{k=1}^{\infty} 1/2^{N_k - (1 - (\alpha - \varepsilon))} \cdot 2^k < \infty.
\]

Hence, with \( \epsilon_1 \equiv (1/k^{(1 - (\alpha + \varepsilon))}_k) \) and \( \epsilon_2 \equiv (1/k^{(1 - (\alpha - \varepsilon))}_k) \), Theorem 1 implies that \( E \) is a set of uniqueness for \( \mathcal{M}_{\epsilon_1} \equiv \mathcal{M}_{(\alpha + \varepsilon)} \) and a set of multiplicity for \( \mathcal{M}_{\epsilon_2} \equiv \mathcal{M}_{(\alpha - \varepsilon)} \).
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3. Proof of Theorem 2. Let \( E \) be a closed set of measure zero and \( \gamma_n \) the number of dyadic intervals which are not disjoint with \( E \). Hence we have

\[
\lim_{n \to \infty} \text{meas } E_n = \lim_{n \to \infty} \gamma_n / 2^n = 0.
\]

Let \( m_\mu \) be a dyadic measure which satisfies (2)(i) and (ii). Then \( m_\mu \) satisfies

\[
\sum_{k=1}^{2^n-1} \varepsilon_k |\hat{\mu}(k)|^2 \geq \theta \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2
\]

\[
= \theta \sum_{n=0}^N \varepsilon_{2^n} \left( \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 - \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right)
\]

\[
= \theta \left\{ \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 - \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\}
\]

\[
= \theta \left\{ \sum_{n=0}^{N-1} \left( \varepsilon_{2^n} - \varepsilon_{2^{n+1}} \right) \cdot \sum_{k=0}^{2^{n+1}-1} |\hat{\mu}(k)|^2 \right\}
\]

\[
- \varepsilon_1 \cdot |\hat{\mu}(0)|^2 + \varepsilon_{2^n} \cdot \sum_{k=0}^{2^{n+1}-1} |\hat{\mu}(k)|^2.
\]

From the assumption we have

\[
1 = |m_\mu(I_0^0)| \leq \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|
\]

\[
\leq \left\{ \sum_{p=0}^{2^n-1} |m_\mu(I_n^p)|^2 \right\}^{1/2} \cdot \left( \sum_{p=0}^{2^n-1} \frac{1}{\text{meas } I_n^p} \right)^{1/2} \leq \left\{ \frac{1}{2^n} \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\}^{1/2} \cdot \{\gamma_n\}^{1/2}
\]

\[
= \left\{ \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\}^{1/2} \cdot \{\gamma_n / 2^n\}^{1/2} = \left\{ \sum_{k=0}^{2^n-1} |\hat{\mu}(k)|^2 \right\}^{1/2} \cdot \{\text{meas } E_n\}^{1/2}.
\]

Combining the above two inequalities, we obtain

\[
\sum_{k=0}^{2^n-1} \varepsilon_k |\hat{\mu}(k)|^2 \geq \theta \left\{ \sum_{n=0}^N \varepsilon_{2^n} \sum_{k=2^n}^{2^{n+1}-1} |\hat{\mu}(k)|^2 \right\}
\]

\[
\geq \theta \left\{ \sum_{n=0}^{N-1} \left( \varepsilon_{2^n} - \varepsilon_{2^{n+1}} \right) \cdot \frac{1}{\text{meas } E_n} - \varepsilon_1 + \frac{\varepsilon_{2^n}}{\text{meas } E_{n+1}} \right\}.
\]

If \( E \) satisfies one of the conditions of Theorem 2, then the last formula tends to infinity. Hence we have

\[
\sum_{k=0}^{\infty} \varepsilon_k |\hat{\mu}(k)|^2 = \infty,
\]

that is, \( \mu \) does not belong to \( \mathcal{M}_e \). Then \( E \) is a set of uniqueness for \( \mathcal{M}_e \). The proof is complete.
Theorem 2 solves the problem left open in [3]. Indeed, if $E$ is of $\alpha$-capacity zero, then $E$ is of $\alpha$-Hausdorff measure zero (see [2]). Namely, $E$ satisfies
\[
\liminf_{n \to \infty} \frac{\gamma_n}{2^{n\alpha}} = 0
\]
where $\{\gamma_n\}$ is defined as before. Thus, it is easy to see that
\[
\limsup_{n \to \infty} \frac{1}{2^{1/\alpha}} \cdot \frac{1}{\text{meas } E_n} = 0.
\]
By Theorem 2, it follows that $E$ is a set of uniqueness for $\mathcal{C}_\alpha^+$. In particular, Theorem 2 contains the following corollary.

**Theorem 3.** When $0 < \alpha < 1$, a closed subset is a set of uniqueness for $\mathcal{C}_\alpha^+$ if and only if it is a set of uniqueness for $\mathcal{C}_\alpha$.

**References**


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