PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 2, October 1983

# ON THE BURES DISTANCE AND THE UHLMANN TRANSITION PROBABILITY OF STATES ON A VON NEUMANN ALGEBRA

HIDEKI KOSAKI[1]

ABSTRACT. Simple expressions for the Bures distance and the Uhlmann transition probability of states on a von Neumann algebra are obtained. Based on these expressions, certain properties are immediately derived.

**0. Introduction.** For normal states $\varphi, \psi$ on a von Neumann algebra $\mathfrak{M}$, the Bures distance $D(\varphi, \psi)$ and the Uhlmann transition probability $P(\varphi, \psi)$ are defined as $\inf \|\zeta_\varphi - \zeta_\psi\|$ and $\sup |(\zeta_\varphi | \zeta_\psi)|^2$, respectively. Here, the infimum and the supremum are taken over all vectors $\zeta_\varphi, \zeta_\psi$ satisfying $\varphi = \omega_{\zeta_\varphi}$, $\psi = \omega_{\zeta_\psi}$ in any normal representation Hilbert space of $\mathfrak{M}$. They are related by

$$D(\varphi, \psi)^2 = \varphi(1) + \psi(1) - 2P(\varphi, \psi)^{1/2} \tag{1}$$

(see [**1, 3, 8**]).

Since they are regarded as a "distance" and a "transition probability" between two states in quantum mechanics, they have been investigated by theoretical physicists as well as operator algebraists [**1,–4, 8**].

The purpose of this article is to obtain certain expressions for $P(\varphi, \psi)$ and $D(\varphi, \psi)$. Our expressions clarify their importance in theoretical physics, and (more importantly) immediately show that the decreasing net $P(\varphi|_{\mathfrak{M}_\iota}, \psi|_{\mathfrak{M}_\iota})$ (resp. the increasing net $D(\varphi|_{\mathfrak{M}_\iota}^{1/2}, \psi|_{\mathfrak{M}_\iota}^{1/2}))$ converges to $P(\varphi, \psi$ (resp. $D(\varphi, \psi)))$ whenever an increasing net $\{\mathfrak{M}_\iota\}_{\iota \in I}$ of von Neumann subalgebras generates $\mathfrak{M}$.

**1. Main results.** As in §0, we fix a von Neumann algebra $\mathfrak{M}$ and $\varphi, \psi \in \mathfrak{M}_*^+$ throughout. We now state our main theorem (which will be proved in §2) and derive some consequences. However, usually results for only $P(\varphi, \psi)$ are stated from which the corresponding results for $D(\varphi, \psi)$ can be obtained through (1).

THEOREM 1. *Let $\{\mathfrak{M}_\iota\}_{\iota \in I}$ be an increasing net of von Neumann subalgebras satisfying $(\bigcup_{\iota \in I} \mathfrak{M}_\iota)'' = \mathfrak{M}$. We then have*

$$P(\varphi, \psi)^{1/2} = \inf\left\{ \sum_{k=1}^{N} \varphi(p_k)^{1/2} \psi(p_k)^{1/2} \right\}. \tag{2}$$

Received by the editors November 15, 1982 and, in revised form, January 12, 1983.
1980 *Mathematics Subject Classification.* Primary 46L10.
[1] Supported in part by NSF(MCS-8102158).


0002-9939/83 $1.00 + $.25 per page

*Here the infimum is taken over all families* $P = \{p_1, \ldots, p_N\}$ *consisting of finitely many projections such that* (i) $p_i \perp p_j$ $(i \neq j)$, (ii) $\sum_{k=1}^N p_k = 1$, (iii) $p_1, \ldots, p_N \in \mathfrak{M}_{\iota_0}$ *for some* $\iota_0 \in I$ *(depending upon* $P$*).*

Notice that, if $\mathfrak{N}$ is a von Neumann subalgebra, we have $P(\varphi, \psi) \leq P(\varphi|_{\mathfrak{N}}, \psi|_{\mathfrak{N}})$ from the definition. Also, when $\mathfrak{N}$ is the finite-dimensional abelian von Neumann subalgebra generated by projections $p_1, \ldots, p_N$, $p_i \perp p_j$ $(i \neq j)$, $\sum_{k=1}^N p_k = 1$, $P(\varphi|_{\mathfrak{N}}, \psi|_{\mathfrak{N}})^{1/2}$ is exactly $\sum_{k=1}^N \varphi(p_k)^{1/2}\psi(p_k)^{1/2}$.

When $\mathfrak{M}_\iota = \mathfrak{M}$ in Theorem 1, we have

COROLLARY 2. *For* $\varphi, \psi \in \mathfrak{M}_*^+$, *we have* $P(\varphi, \psi) = \inf\{P(\varphi|_{\mathfrak{N}}, \psi|_{\mathfrak{N}})$: $\mathfrak{N}$ *is a finite-dimensional abelian von Neumann subalgebra*$\}$.

This is actually a "discrete" version of [**2**] and closely related to [7]. Noticing condition (iii) in Theorem 1, we immediately have

COROLLARY 3. *If an increasing net* $\{\mathfrak{M}_\iota\}_{\iota \in I}$ *of von Neumann subalgebras generates* $\mathfrak{M}$, *then the (decreasing) net* $\{P(\varphi|_{\mathfrak{M}_\iota}, \psi|_{\mathfrak{M}_\iota})\}$ *(bounded by* $P(\varphi, \psi)$ *from below) converges to* $P(\varphi, \psi)$.

The following two remarks are in order:

(i) Corollary 2 and (1) mean that

$$D(\varphi, \psi) = \sup\left\{\left[\sum_{k=1}^N \left(\varphi(p_k)^{1/2} - \psi(p_k)^{1/2}\right)^2\right]^{1/2}\right\},$$

where the supremum is taken over all families $P = (p_1, \ldots, p_N)$ consisting of finitely many projections in $\mathfrak{M}$ such that $p_i \perp p_j$ $(i \neq j)$ and $\sum_{k=1}^N p_k = 1$. When a physical system is described by $\mathfrak{M}$, each selfadjoint $x = \int_{-\infty}^{\infty} \lambda \, de(\lambda)$ in $\mathfrak{M}$ is considered as an observable. Then, for a state $\varphi \in \mathfrak{M}_*^+$ and a partition $\mathbf{R} = \bigcup_{k=1}^N E_k$ (of $\mathbf{R}$ into disjoint Borel subsets), $\varphi(p_k) = \int_{E_k} d_\varphi(e(\lambda))$ (with $p_k = \int_{E_k} de(\lambda)$) is interpreted as the probability that a measurement of $x$ performed on the system in the state $\varphi$ yields a result lying in $E_k$. Thus, $D(\varphi, \psi) \leq \varepsilon$ for a small $\varepsilon > 0$ means that two states $\varphi, \psi$ give almost similar measurements for any observable $x$ in the sense that $\sum_{k=1}^N (\varphi(p_k)^{1/2} - \psi(p_k)^{1/2})^2 \leq \varepsilon^2$ (for any partition $\mathbf{R} = \bigcup_{k=1}^N E_k$). Therefore, the Bures distance is quite suitable to describe a distance between two (physical) states.

(ii) When $\mathfrak{A}$ is the UHF algebra defined by an increasing sequence of finite-dimensional factors $\mathfrak{A}_n$, $n = 1, 2, \ldots$, for two states $\varphi, \psi \in \mathfrak{A}_+^*$, $\lim_{n \to \infty} P(\varphi|_{\mathfrak{A}_n}, \psi|_{\mathfrak{A}_n})$ is exactly the Uhlmann transition probability between the normal extensions $\tilde{\varphi}$, $\tilde{\psi} \in (\mathfrak{A}^{**})_*^+$ on the universal enveloping von Neumann algebra $\mathfrak{A}^{**}$ [**6**]. Notice that $\lim_{n \to \infty} P(\varphi|_{\mathfrak{A}_n}, \psi|_{\mathfrak{A}_n})$ can be computed at the matrix algebra level. (This is especially easy when each $\mathfrak{A}_n$ is a tensor product, and $\varphi, \psi$ are product states.)

**2. Proof of Theorem 1.** Obviously the right-hand side of (2) is jointly monotone and upper semicontinuous in $(\varphi, \psi) \in \mathfrak{M}_*^+ \times \mathfrak{M}_*^+$, while the map $(\varphi, \psi) \mapsto P(\varphi, \psi)^{1/2}$ is known to be jointly monotone and continuous. Hence, we may and do assume that $\varphi \leq l\psi$ for some $l > 0$ (by approximating $\psi$ by $\psi + n^{-1}\varphi$, $n = 1, 2, \ldots$, from above). Then we may also assume $\|\psi\| \leq 1$ and $\psi$ is faithful. Using the GNS construction, we can now assume that $\mathfrak{M}$ acts on a Hilbert space $\mathcal{H}$ and $\psi = \omega_{\xi_0}$ with some vector $\xi_0 \in \mathcal{H}$ ($\|\xi_0\| = \psi(1)^{1/2} \leqslant 1$).

Due to $\varphi \leqslant l\psi$, there exists a positive operator $h \in \mathfrak{M}$ such that $\varphi = \omega_{h\xi_0}$, $\|h\| \leqslant l^{1/2}$ (Sakai's Radon-Nikodym derivative [5]). It is known that $P(\varphi, \psi)^{1/2} = (h\xi_0 | \xi_0)$. Thus (due to the remark before Corollary 2) it suffices to show that

$$(2)' \qquad (h\xi_0 | \xi_0) \geqslant \inf\left\{ \sum_{k=1}^{n} \varphi(p_k)^{1/2} \psi(p_k)^{1/2} \right\}$$

(under the above hypotheses).

We choose and fix a small $\varepsilon > 0$. Kaplansky's density theorem [5] guarantees the existence of a positive $\tilde{k}$ in $\bigcup_{\iota \in I} \mathfrak{M}_\iota$ (hence $\tilde{k} \in \mathfrak{M}_{\iota_0}$ for some $\iota_0 \in I$) satisfying $\|(h - \tilde{k})\xi_0\| \leqslant \varepsilon$ and $\|\tilde{k}\| \leqslant \|h\| \leqslant l^{1/2}$. Introducing the invertible positive operator $k = \tilde{k} + \mathrm{Min}(\varepsilon, l^{1/2})1 \in \mathfrak{M}_{\iota_0}^+$, we observe that

$$(3) \qquad \|(h - k)\xi_0\| \leqslant 2\varepsilon, \qquad \|k\| \leqslant 2l^{1/2}.$$

The obvious estimate $|((h - k)\xi_0 | \xi_0)| \leqslant 2\varepsilon$ implies

$$(4) \qquad (h\xi_0 | \xi_0) + 2\varepsilon \geqslant (k\xi_0 | \xi_0).$$

Let $k = \int_0^{\|k\|} \lambda \, de(\lambda)$ be the spectral decomposition so $(k\xi_0 | \xi_0) = \int_0^{\|k\|} \lambda \, d\psi(e(\lambda))$. (Notice that $e(0) = \text{strong-lim}_{\lambda \searrow 0} e(\lambda) = 0$.) For each $n \in \mathbf{N}_+$ (independent of $\varepsilon$), we set

$$e_{i,n} = e(i\|k\|n^{-1}) - e((i-1)\|k\|n^{-1}), \qquad i = 1, 2, \ldots, n,$$

so $e_{i,n} \perp e_{j,n}$ $(i \neq j)$, $\sum_{i=1}^n e_{i,n} = 1$, and they are projections in $\mathfrak{M}_{\iota_0}$. Due to the estimates

$$\sum_{i=1}^{n} i\|k\|n^{-1}\psi(e_{i,n}) - \int_0^{\|k\|} \lambda \, d\psi(e(\lambda)) \leqslant \|k\|n^{-1} \int_0^{\|k\|} d\psi(e(\lambda))$$
$$= \|k\|n^{-1}\psi(1) \leqslant 2l^{1/2}n^{-1} \quad \text{(recall (3) and } \psi(1) \leqslant 1\text{)},$$

(4) implies

$$(5) \qquad (h\xi_0 | \xi_0) + 2\varepsilon + 2l^{1/2}n^{-1} \geqslant \sum_{i=1}^{n} i\|k\|n^{-1}\psi(e_{i,n}).$$

We now set $\tilde{\psi} = \omega_{k\xi_0} \in \mathfrak{M}_*^+$ and observe that

$$\tilde{\psi}(e(\mu)) = \psi(ke(\mu)k) = \psi\left(\int_0^{\mu} \lambda^2 \, de(\lambda)\right) = \int_0^{\mu} \lambda^2 \, d\psi(e(\lambda)).$$

Since $\lambda^2$ is increasing, we have

$$\tilde{\psi}(e_{i,n}) \leqslant \{i\|k\|n^{-1}\}^2 \psi(e_{i,n}).$$

Taking the square roots, multiplying by $\psi(e_{i,n})^{1/2}$, and summing over $i$, we get

$$\sum_{i=1}^{n} \tilde{\psi}(e_{i,n})^{1/2} \psi(e_{i,n})^{1/2} \leqslant \sum_{i=1}^{n} i\|k\|n^{-1}\psi(e_{i,n}).$$

Hence (5) yields

$$(h\xi_0|\xi_0) + 2\varepsilon + 2l^{1/2}n^{-1} \geqslant \sum_{i=n}^{n} \tilde{\psi}(e_{i,n})^{1/2}\psi(e_{i,n})^{1/2}. \tag{6}$$

We now notice that, for $\alpha, \beta \geqslant 0$, we have $(\alpha^{1/2} - \beta^{1/2})^2 \leqslant |\alpha - \beta|$. This fact and the Cauchy-Schwarz inequality imply that

$$\begin{aligned}\delta &= \left|\sum_{i=1}^{n} \tilde{\psi}(e_{i,n})^{1/2}\psi(e_{i,n})^{1/2} - \sum_{i=1}^{n} \varphi(e_{i,n})^{1/2}\psi(e_{i,n})^{1/2}\right|^2 \\ &\leqslant \left[\sum_{i=1}^{n} \psi(e_{i,n})\right]\left[\sum_{i=1}^{n} \left(\tilde{\psi}(e_{i,n})^{1/2} - \varphi(e_{i,n})^{1/2}\right)^2\right] \\ &\leqslant \psi(1)\left[\sum_{i=1}^{n} |\tilde{\psi}(e_{i,n}) - \varphi(e_{i,n})|\right] \leqslant \sum_{i=1}^{n} |(\tilde{\psi} - \varphi)(e_{i,n})|.\end{aligned}$$

Since $\tilde{\psi} - \varphi \in \mathfrak{M}_*$ is selfadjoint, the positive part $|\tilde{\psi} - \varphi|$ of its polar decomposition [5] enjoys $|(\tilde{\psi} - \varphi)(x)| \leqslant |\tilde{\psi} - \varphi|(x)$ for any $x \in \mathfrak{M}_+$. We thus further estimate

$$\begin{aligned}\delta &\leqslant \sum_{i=1}^{n} |\tilde{\psi} - \varphi|(e_{i,n}) = |\tilde{\psi} - \varphi|(1) = \|\tilde{\psi} - \varphi\| = \|\omega_{k\xi_0} - \omega_{h\xi_0}\| \\ &\leqslant \|(k-h)\xi_0\|\{\|k\| + \|h\|\}\|\xi_0\| \leqslant 2\varepsilon\{2l^{1/2} + l^{1/2}\} = 6\varepsilon l^{1/2}\end{aligned}$$

due to (3).

Therefore, (6) implies

$$(h\xi_0|\xi_0) + 2\varepsilon + 2l^{1/2}n^{-1} + 6^{1/2}\varepsilon^{1/2}l^{1/4} \geqslant \sum_{i=1}^{n} \varphi(e_{i,n})^{1/2}\psi(e_{i,n})^{1/2}.$$

Since $\varepsilon > 0$ is arbitrary and $n \in \mathbf{N}_+$ is independent of $\varepsilon$, this estimate means (2)′. Q.E.D.

## References

1. H. Araki, *Positive cones, Radon-Nikodym theorems, relative Hamiltonian and Gibbs condition in statistical mechanics* ($C^*$-Algebras and their Applications to Statistical Mechanics and Quantum Field Theory (D. Kastler, ed.), North-Holland, Amsterdam, 1976.

2. H. Araki and G. Raggio, *A remark on transition probability*, Lett. Math. Phys. **6** (1982), 237–240.

3. D. Bures, *An extension of Kakutani's theorem on infinite product measures to the tensor product of semifinite von Neumann algebras*, Trans. Amer. Math. Soc. **135** (1969), 199–212.

4. S. Gudder, J. P. Marchand and W. Wyss, *Bures distance and relative entropy*, J. Math. Phys. **20** (1979), 1963–1966.

5. J. Dixmier, *Les algèbres d'opérateurs dans l'espace Hilbertien*, Gauthier-Villars, Paris, 1969.

6. ______, *Les $C^*$-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1969.

7. N. Hadjisavvas, *On Cantoni's generalized transition probability*, Comm. Math. Phys. **83** (1982), 43–48.

8. A. Uhlmann, *The "transition probability" in the state space of a *-algebra*, Rep. Math. Phys. **9** (1976), 273–279.

Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

*Current address*: Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104-3859