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A NOTE ON α-COMPACT SPACES

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ABSTRACT. For an infinite cardinal α, $m(\alpha)$ denotes the least measurable cardinal, if one exists, not less than α. We give easy proofs of generalizations of some results on realcompact spaces. Among these we prove the following generalization of a theorem of A. Kato.

Let $\{X_i: i \in I\}$ be a collection of spaces each having at least two elements. Then the k-box product $(\prod X_i)_k$ is α-compact if and only if either X_i is α-compact for each $i \in I$ and $k \leqslant m(\alpha)$ or $|I| < m(\alpha)$.

All spaces discussed in this paper are completely regular Hausdorff spaces. Let X be a space and α an uncountable cardinal. Then $Z(X)$ denotes the set of all zero sets in X and βX the Stone-Čech compactification of X. A family $\mathfrak{A}$ of sets has the α-intersections property (α.i.p. for short) if $\bigcap \mathfrak{B} \neq \varnothing$ whenever $\mathfrak{B} \subseteq \alpha$ and $|\mathfrak{B}| < \alpha$. We write $\beta_\alpha(X) = \{p \in \beta X: p \text{ has the } \alpha\text{.i.p.}\}$. The space X is said to be α-compact if $X = \beta_\alpha X$. See [**1** and **4**] for a discussion of these spaces and for references to other papers on the subject.

Let k be an uncountable cardinal and let (X, τ) be a space. $(X, \tau(k))$ denotes the space with basis the family of all intersections of less than k members of τ. Let $\{X_i: i \in I\}$ be a family of spaces. Then $(\prod_{i \in I} X_i)_k$ denotes the k-box product of the family. This space has as basis sets of the form $\prod U_i$ where $U_i \neq X_i$ for less than k many i and U_i is open in X_i.

Let α be an uncountable cardinal number. Then $m(\alpha)$ stands for the least measurable cardinal such that $\alpha \leqslant m(\alpha)$ (if one exists). We say that m is measurable if there is a discrete space A with $|A| = m$ and $\beta_m(A) - A \neq \varnothing$.

THEOREM 1. *Let $p \in \beta_\alpha(X) - X$ and $\mathfrak{B} \subseteq p$ such that*
(i) $\bigcap \mathfrak{B} = \varnothing$, *and*
(ii) $\mathfrak{B}' \subseteq \mathfrak{B}$ *implies* $\bigcap \mathfrak{B}' \in Z(X)$.
Then $|\mathfrak{B}| \geqslant m(\alpha)$.

PROOF. We may assume that $\mathfrak{B}$ has the minimum cardinality with respect to having empty intersection. Let $|\mathfrak{B}| = m$. Then $\mathfrak{B}$ has the m-i.p. In fact if $\mathfrak{B}' \subseteq \mathfrak{B}$ and $|\mathfrak{B}'| < \alpha$ then $\bigcap \mathfrak{B}' \in p$. If this were not the case then there would exist $Z \in p$ such that $Z \cap (\bigcap \mathfrak{B}') = \varnothing$. Hence the family $\mathfrak{B}' \cup \{Z\}$ would be a subset of p with empty intersection, contrary to the minimality of $|\mathfrak{B}|$.

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Let $\mathfrak{B} = \{B_a\colon a \in A\}$ with $|A| = |\mathfrak{B}|$. For $D \subseteq A$ let $Z_D = \bigcap_{a \in D} B_a$ and let $q = \{D \subseteq A\colon Z_D \notin p\}$. Then

(i) $Z_\varnothing = X \in p \Rightarrow \varnothing \notin q$; $Z_A = \varnothing \notin p \Rightarrow A \in q$.

(ii) $D \notin q \Rightarrow Z_D \in p$. Then $Z_D \cap Z_{A-D} = Z_{D \cup (A-D)} = Z_A \notin p$. Hence $Z_{A-D} \notin p$. which implies that $A - D \in q$.

(iii) $D \subseteq E$ and $D \in q \Rightarrow Z_D \notin p$ and since $Z_E \subseteq Z_D$ we have $Z_E \notin p$. Hence $E \in q$. From (i), (ii) and (iii) we conclude that q is an ultrafilter on A. Since $B_a \in p$ for any a we see that $\{a\} \notin q$. Hence q is free.

(iv) Let $\{D_i\colon i \in I\} \subseteq q$ where $|I| < m$. Then $D_i \in q \Rightarrow A - D_i \notin q \Rightarrow Z_{A-D_i} \in p$ for all $i \in I \Rightarrow \bigcap Z_{A-D_i} \in p \Rightarrow Z_{\bigcup(A-D_i)} \in p \Rightarrow \bigcup(A - D_i) = A - \bigcap D_i \notin q \Rightarrow \bigcap D_i \in q$.

Thus q has the m-i.p. Hence m is measurable and $m \geqslant \alpha$. Hence $m \geqslant m(\alpha)$ which implies that $|\mathfrak{B}| \geqslant m(\alpha)$ as claimed.

COROLLARY 1. *A discrete space A is α-compact if and only if $|A| < m(\alpha)$.*

PROOF. If $|A| \geqslant m(\alpha)$ then clearly $\beta_m(A) - A \neq \varnothing$ and so A is not α-compact. Conversely, if $p \in \beta_\alpha(A) - A$ then with $\beta = p$ in Theorem 1 we see that $|p| \geqslant m(\alpha)$. Hence $2^{2^{|A|}} \geqslant m(\alpha)$ and since $m(\alpha)$ is strongly inaccessible we have $|A| \geqslant m(\alpha)$.

COROLLARY 2. *A metric space M is α-compact if and only if $|M| < m(\alpha)$.*

PROOF. Let M be α-compact. Then every closed discrete subset (being α-compact) has cardinality less than $m(\alpha)$ by Corollary 1. It is well known that there are closed discrete subsets D_n, $n = 1, 2, \ldots$, such that $D = \bigcup D_n$ is dense in M. Hence $|D| < m(\alpha)$ and so $|M| \leqslant |D|^\omega < m(\alpha)$.

Conversely, suppose M is not α-compact. Let $p \in \beta_\alpha(M) - M$. Put $\mathfrak{B} = p$ in Theorem 1. We have $|\mathfrak{B}| \geqslant m(\alpha)$, concluding the proof.

A space is said to be topologically complete if it is homeomorphic to a closed subspace of a product of metric spaces. The following is a generalization of the Katetov-Shirota theorem which states that a topologically complete space is real compact if and only if each closed discrete subspace has cardinality less than the first uncountable measurable cardinal. Real compact spaces are α-compact spaces for $\alpha = \omega^+$.

COROLLARY 3. *A topologically complete space is α-compact if and only if each closed discrete subset has cardinality less than $m(\alpha)$.*

PROOF. ($\Rightarrow$) This implication is trivial.

($\Leftarrow$) Let X be a closed subspace of a product of metric spaces $\Pi_{i \in I} M_i$ such that each closed discrete subset has cardinality less than $m(\alpha)$, we may assume that X projects onto M_i for $i \in I$. Then since a closed discrete subset of M_i may be regarded as the projection of a closed discrete subset of X we see that each closed discrete subset of M_i has cardinality less than $m(\alpha)$ for each i and so X is α-compact.

A family $\mathfrak{F}$ of zero sets of X is called an α-base if to each $Z \in Z(X)$ and each $x \in Z$ there is $\mathfrak{B} \subseteq \mathfrak{F}$ with

(i) $x \in \bigcap \mathfrak{B} \subseteq Z$,

(ii) $|\mathfrak{B}| < m(\alpha)$,

(iii) $\bigcap \mathfrak{B}' \in Z(X)$ for all $\mathfrak{B}' \subseteq \mathfrak{B}$.

EXAMPLE 1. Let (X, τ) be a space and $w \leqslant \kappa \leqslant m(\alpha)$. Then $Z(X)$ is an α-base for $(X, \tau(\kappa))$.

EXAMPLE 2. (Cf. [**4**, Lemma 3.9].) The family $\bigcup\{\pi_i^{-1}(Z(X_i))\colon i \in I\}$ where π_i is the projection of $X = (\prod X_i)_\kappa$ to X_i is an α-base for X if $\omega \leqslant \kappa \leqslant m(\alpha)$.

THEOREM 2. *Suppose $f\colon X \to Y$ is a continuous surjection such that $f^{-1}(Z(Y))$ is an α-base for X. If Y is α-compact then so is X.*

PROOF. Suppose X is not α-compact, and let $p \in \beta_\alpha(X) - X$. Let $\bar{f}(p) = y$ where $\bar{f}$ is the Stone-Čech extension of f. Let $f(x) = y$. Then there is $Z \in Z(X)$ such that $x \in Z \notin p$. Since $f^{-1}(Z(Y))$ is an α-base there is $\mathfrak{B} \subseteq f^{-1}(Z(Y))$ such that $x \in \bigcap \mathfrak{B} \subseteq Z$, $|\mathfrak{B}| < m(\alpha)$ and $\mathfrak{B}' \subseteq \mathfrak{B} \Rightarrow \bigcap \mathfrak{B}' \in Z(X)$. Clearly $\mathfrak{B} \subseteq p$. There is $\bar{Z} \in p$ such that $\bar{Z} \cap (\bigcap \mathfrak{B}) = \varnothing$. Hence the family $\{\bar{Z}\} \cup \mathfrak{B}$ is a subset of p with empty intersection. By Theorem 1, $|\mathfrak{B}| \geqslant m(\alpha)$ contrary to the assumption that $|\mathfrak{B}| < m(\alpha)$. Hence X is α-compact as was to be proved.

COROLLARY 4. *Let (X, τ) be α-compact and let $\omega \leqslant \kappa \leqslant m(\alpha)$. Then $(X, \tau(\kappa))$ is α-compact.*

PROOF. The identity map from $(X, \tau(\kappa))$ to (X, τ) satisfies the conditions of Theorem 2.

COROLLARY 5. *If $\{X_i\colon i \in I\}$ is a collection of spaces with $|X_i| \geqslant 2$ for each i then $(\prod X_i)_\kappa$ is α-compact if and only if X_i is α-compact for each $i \in I$ and $\kappa \leqslant m(\alpha)$ or $|I| < m(\alpha)$.*

PROOF. Suppose $X = (\prod X_i)_\kappa$ is α-compact. Then for each $i \in I$, X_i can be considered as a closed subspace of X and so is α-compact. Let $J \subseteq I$ and $|J| < \kappa$. Let $D_j \subseteq X_j$ have exactly two elements for each j in J and let D_j be a singleton subset of X_j for $j \notin J$. Then $\prod D_j$ is a closed discrete subset of X and so has cardinality less than $m(\alpha)$. Hence $\kappa \leqslant m(\alpha)$ or $|I| < m(\alpha)$.

Conversely suppose X_i is α-compact for each $i \in I$ and $\kappa \leqslant m(\alpha)$ or $|I| < m(\alpha)$. Then $\prod_{i \in I} X_i$ is α-compact and $Z(\prod X_i)$ is an α-base for $(\prod X_i)_\kappa$. Hence $(\prod X_i)_\kappa$ is α-compact by Theorem 2.

Corollary 5 is a generalization of Theorem 2.4 of Kato [**3**] and Corollary 4 is a generalization of 3.1 of the same paper. Kato deals with realcompact spaces. His method is quite different from ours.

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