CARDINAL FUNCTIONS
ON PIXLEY-ROY HYPERSPACES

SHOZO SAKAI

Abstract. Some cardinal functions on the Pixley-Roy hyperspace \( \mathcal{F}[X] \) of a space \( X \) are determined by those on \( X \), and conditions of \( X \) by cardinal functions from which \( \mathcal{F}[X] \) has or has not some properties, e.g. cosmic, paracompact, ccc, etc. are given.

All spaces considered in this paper will be assumed to be infinite \( T_1 \)-spaces. Cardinals are the initial ordinals, \( \tau \) will always denote an infinite cardinal and \( \omega \) is the smallest infinite cardinal. The cardinality of a set \( X \) is denoted by \( |X| \) and \( c \) is the cardinality of the continuum. The successor cardinal of \( \tau \) is denoted by \( \tau^+ \).

The Pixley-Roy hyperspace \( \mathcal{F}[X] \) of a space \( X \) has as its underlying set the collection of all finite nonempty subsets of \( X \). If \( A \in \mathcal{F}[X] \), then the basic nbds (= neighbourhoods) of \( A \) are \( [A, U] = \{ S \in \mathcal{F}[X] : A \in S \subseteq U \} \), where \( U \) is an open set of \( X \) containing \( A \).

We refer to [J] for the following cardinal functions—\( w \) (weight), \( \pi \) (\( \pi \)-weight), \( nw \) (net weight), \( d \) (density), \( c \) (cellularity), \( L \) (Lindelöf-degree), \( p \) (= e (extent) in [E]), \( \chi \) (character), \( \psi \) (pseudo character), \( \Psi \) (\( \Psi(X) = \omega \) iff \( X \) is perfect), \( \psi_\Delta \) (diagonal degree), \( \pi\chi \) (\( \pi \)-character), \( t \) (tightness) and \( psw \) (point separating weight). However, following Engelking [E], the hereditary density and the hereditary Lindelöf-degree are denoted by \( hd \) and \( hL \), respectively.

In the following, Theorem 2 is the main result of this paper. Some results in it were given by van Douwen [vD], Lutzer [L] and Przymusiñski [P]. Though Proposition 1 is an auxiliary one for Theorem 2, it is interesting in itself.

Proposition 1. If \( A \in \mathcal{F}[X] \), then the following hold:

1. \( \psi(A, \mathcal{F}[X]) = \max\{\psi(x, X) : x \in A\} \).
2. \( \pi\chi(A, \mathcal{F}[X]) = \chi(A, \mathcal{F}[X]) \leq \max\{\chi(x, X) : x \in A\} \).

If \( x \in X \), then the following hold:

3. \( \chi(\{x\}, \mathcal{F}[X]) = \chi(x, X) \).
4. \( t(x, X) \leq t(\{x\}, \mathcal{F}[X]) \).

If \( X \) is Hausdorff and \( A \in \mathcal{F}[X] \), then the following holds:

5. \( \chi(A, \mathcal{F}[X]) = \max\{\chi(x, X) : x \in A\} \).

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Proof. (1) Let \( A = \{x_1, \ldots, x_n\} \subseteq \mathbb{T}[X] \). For each \( i \in \{1, \ldots, n\} \), there is a local \( \psi \)-base \( \mathcal{U}_i \) of \( x_i \) in \( X \) such that \( |\mathcal{U}_i| \leq \psi(x_i, X) \). Let

\[ \mathcal{U} = \{ \bigcup \{ U_i : i = 1, \ldots, n \} : U_i \in \mathcal{U}_i \} \quad \text{and} \quad \mathcal{B} = \{ [A, U] : U \in \mathcal{U} \}. \]

Then \( \mathcal{B} \) is a local \( \psi \)-base of \( A \) in \( \mathbb{T}[X] \) and \( |\mathcal{B}| \leq \max\{\psi(x_i, X) : i = 1, \ldots, n\} \). Thus, we have \( \psi(A, \mathbb{T}[X]) \leq \max\{\psi(x, X) : x \in A\} \). Conversely, let \( \mathcal{B} \) be a local \( \psi \)-base of \( A \) in \( \mathbb{T}[X] \) and \( x \in A \). Then \( \mathcal{U} = \{ B \setminus (A \setminus \{x\}) : [A, B] \in \mathcal{B} \} \) is a local \( \psi \)-base of \( x \) in \( X \). Therefore, we have \( \psi(x, X) \leq \psi(A, \mathbb{T}[X]) \) and \( \max\{\psi(x, X) : x \in A\} \leq \psi(A, \mathbb{T}[X]) \).

(2) Let \( \mathcal{B} \) be a local \( \pi \)-base of \( A \) in \( \mathbb{T}[X] \). For each \( B \in \mathcal{B} \), we can take an element \( A_B \subseteq B \) and an open nbhd \( U_B \) of \( A_B \) in \( X \) such that \( [A_B, U_B] \subseteq B \). Then, \( \mathcal{U} = \{ [A, U_B] : B \in \mathcal{B}, A \subseteq A_B \} \) is a nbhd base of \( A \) in \( \mathbb{T}[X] \). For, let \( V \) be an open nbhd of \( A \) in \( X \).

Then, there is \( B \in \mathcal{B} \) such that \( B \subseteq [A, V] \). Therefore, \( [A_B, U_B] \subseteq B \subseteq [A, V] \), i.e. \( A \subseteq A_B \subseteq U_B \subseteq V \). Thus, \( [A_B, U_B] \subseteq [A, V] \) and \( \mathfrak{B} \) is a nbhd base of \( A \) in \( \mathbb{T}[X] \). Since \( |\mathcal{U}| \leq |\mathcal{B}| \), we have \( \chi(A, \mathbb{T}[X]) \leq \pi\chi(A, \mathbb{T}[X]). \)

(3) The proof is clear.

(4) Let \( H \subseteq X \) and \( x \in X \setminus H \). Let us put \( \mathcal{K} = \mathbb{T}[H \cup \{x\}] \\setminus \{\{x\}\} \).

Then \( \{x\} \subseteq \mathcal{K} \subseteq X \). Therefore, there is a subset \( \mathcal{M} \subseteq \mathcal{K} \) such that \( |\mathcal{M}| \leq t(x, \mathbb{T}[X]) \) and \( \{x\} \subseteq \mathcal{M} \subseteq X \). Let \( \mathcal{M} = \{ \{x\} \} \cup \mathcal{K} \subseteq X \). Then \( x \in cl_X \mathcal{M} \). For, let \( V \) be a nbhd of \( x \) in \( X \).

Then, there is \( \mathcal{V} \subseteq \mathcal{M} \) such that \( \mathcal{V} \subseteq \{x\} \). Since \( x \notin \mathcal{V} \), there is an element \( x \in \mathcal{V} \cap \mathcal{M} \). Then, \( x \in cl_X \mathcal{M} \).

Since \( \mathcal{M} \subseteq H \) and \( |\mathcal{M}| \leq |\mathcal{K}| \leq t(x, \mathbb{T}[X]) \), we have \( t(x, X) \leq t(x, \mathbb{T}[X]) \).

(5) Let us assume that there is \( x \in A \) with \( \chi(A, \mathbb{T}[X]) < \chi(x, X) \). Let \( \tau = \mathfrak{U}(A, \mathbb{T}[X]) \) and \( \mathfrak{U} \) a nbhd base of \( A \) in \( \mathbb{T}[X] \) with \( |\mathfrak{U}| \leq \tau \). We can assume \( \mathfrak{U} = \{ [A, U] : \alpha < \tau \} \), where \( U \) is an open set of \( X \) containing \( A \). Since \( X \) is Hausdorff, there is an open nbhd \( U \) of \( x \) with \( (A \setminus \{x\}) \cap (\bigcup \{U \cap \alpha \} = \varnothing \). Since \( \chi(x, X) \tau \), there is an open nbhd \( V \) of \( x \) with \( U \cap U \subset V \) for each \( \alpha < \tau \). Let \( W = V \cup (X \setminus cl_X U) \).

Then \( A \subseteq W \) is an open nbhd of \( A \) in \( \mathbb{T}[X] \). Since \( U \subseteq W \), \( [A, U] \subseteq [A, W] \) for each \( \alpha < \tau \). Therefore, \( \mathfrak{U} \) is not a nbhd base of \( A \), a contradiction. Thus, we have \( \chi(x, X) \leq \chi(A, \mathbb{T}[X]) \) for each \( x \in A \). This completes the proof of Proposition 1.

Theorem 2. For each space \( X \), the following hold:

(1) \( |\mathbb{T}[X]| = \text{nw}(\mathbb{T}[X]) = \text{d}(\mathbb{T}[X]) = L(\mathbb{T}[X]) = p(\mathbb{T}[X]) = |X| \).

(2) \( \chi(\mathbb{T}[X]) = \pi\chi(\mathbb{T}[X]) = \chi(X) \).

(3) \( w(\mathbb{T}[X]) = \pi(\mathbb{T}[X]) = \chi(X) |X| \).

(4) \( \text{psw}(\mathbb{T}[X]) = \psi(\mathbb{T}[X]) = \psi(X) \).

(5) \( \psi(X) = \omega \) iff \( \Psi(\mathbb{T}[X]) = \omega \) iff \( \psi(\mathbb{T}[X]) = \omega \).

(6) \( \text{hd}(X)h L(X) \leq c(\mathbb{T}[X]) \leq \text{nw}(X) \).

Proof. (1) Since \( \{\{x\} : x \in X\} \) is a closed discrete subspace of \( \mathbb{T}[X] \), \( |X| \leq p(\mathbb{T}[X]) \).

Therefore, (1) is clear.

(2) By Proposition 1, this is trivial.

(3) From

\[ \chi(X) = \pi\chi(\mathbb{T}[X]) \leq \pi(\mathbb{T}[X]) \quad \text{and} \quad |X| = \text{d}(\mathbb{T}[X]) \leq \pi(\mathbb{T}[X]), \]

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we have $\chi(X) |X| \leq \pi(\mathcal{F}[X]) \leq w(\mathcal{F}[X])$. On the other hand,

$$w(\mathcal{F}[X]) \leq \chi(\mathcal{F}[X]) |\mathcal{F}[X]|$$

is always true. Therefore, by (1) and (2), (3) is proved.

(4) Since $\psi(\mathcal{F}[X]) \leq psw(\mathcal{F}[X])$ is well known, we will prove $psw(\mathcal{F}[X]) \leq \psi(X)$ only. For each $A \in \mathcal{F}[X]$, there is a local $\psi$-base $\mathcal{B}(A)$ of $A$ with $|\mathcal{B}(A)| \leq \psi(\mathcal{F}[X]) = \psi(X)$. Then, $\mathcal{B} = \{[A, U] : A \in \mathcal{F}[X], U \in \mathcal{B}(A)\}$ is a pseudo base of $\mathcal{F}[X]$. Let $S \in \mathcal{F}[X]$, $S \subseteq [A, U]$ if $A \subset S \subset A$. Since $|S| < \omega$, $\{A : \emptyset \neq A \subset S\}$ is finite. Therefore, $\text{ord}(S, \mathcal{B}) = \{([A, U] \in \mathcal{B} : S \subseteq [A, U]) \leq \psi(X)$. Thus we have $\text{ord}(\mathcal{B}) = \sup(\text{ord}(S, \mathcal{B}) : S \in \mathcal{F}[X]) \leq \psi(X)$. Therefore, $psw(\mathcal{F}[X]) \leq \psi(X)$.

(5) $\psi(X) = \omega$ if $\psi(\mathcal{F}[X]) = \omega$ appears in [L] and [P], and $\psi(\mathcal{F}[X]) \leq \psi_{\omega}(\mathcal{F}[X])$ is well known. Therefore, only $\psi_{\omega}(\mathcal{F}[X]) \leq \psi(\mathcal{F}[X]) = \omega$ needs proof. Let $\psi(\mathcal{F}[X]) = \omega$ and $\mathcal{B}(A) = \{U_n(A) : n < \omega\}$ be a decreasing countable local $\psi$-base of each $A \in \mathcal{F}[X]$. We can assume that $U_n(A)$ is of the form $[A, V]$, where $V$ is an open nbd of $A$ in $X$. Since $B_n = \cup \{U_n(A) \times U_n(A) : A \in \mathcal{F}[X]\}$ is an open nbd of the diagonal $\Delta(\mathcal{F}[X]) = \{(A, A) : A \in \mathcal{F}[X]\}$ in $\mathcal{F}[X]^2 = \mathcal{F}[X] \times \mathcal{F}[X]$, $\mathcal{B} = \{B_n : n < \omega\}$ is a countable local $\psi$-base of $\Delta(\mathcal{F}[X])$ in $\mathcal{F}[X]^2$. For, let $(S, T) \in \mathcal{F}[X]^2 \setminus \Delta(\mathcal{F}[X])$. We may assume that there is an element $t \in T \setminus S$. For each nonempty $A \subseteq S$, there exists $n(A) < \omega$ such that $\{t\} \notin \cup \{U_n(A) : A \in \mathcal{F}[X]\}$. If we put $n_0 = \max\{n(A) : \emptyset \neq A \subseteq S\}$, then $T \notin \cup \{U_n(A) : \emptyset \neq A \subseteq S\}$ for each $n \geq n_0$. Therefore, $(S, T) \notin B_{n_0}$ and $\mathcal{B}$ is a local $\psi$-base of $\Delta(\mathcal{F}[X])$ in $\mathcal{F}[X]^2$. Therefore, we have $\psi_{\omega}(\mathcal{F}[X]) = \omega$.

(6) Since $c(\mathcal{F}[X]) \leq nw(X)$ appears in [L], it suffices to show $hd(X) hL(X) \leq c(\mathcal{F}[X])$.

Let us assume $hL(X) > \tau$. Then, there are a subspace $Y$ of $X$ and an open covering $\mathcal{G}$ of $Y$ such that $Y \subseteq \cup \mathcal{G}$ for each subfamily $\mathcal{G}'$ of $\mathcal{G}$ with $|\mathcal{G}'| \leq \tau$. We can assume that each element of $\mathcal{G}$ is open in $X$. By transfinite induction, we can easily construct $\{x_\alpha : \alpha < \tau^+\} \subseteq Y$ and $\{G_\alpha : \alpha < \tau^+\} \subseteq \mathcal{G}$ with $x_\alpha \in G_\alpha \setminus \bigcup \{G_\beta : \beta < \alpha\}$ for each $\alpha < \tau^+$. If $\beta < \alpha < \tau^+$, since $x_\alpha \notin G_\beta$, we have $\{x_\beta, G_\beta \setminus \{x_\alpha, G_\alpha\} = \emptyset$. Thus, $\{\{x_\alpha, G_\alpha : \alpha < \tau^+\}$ is a pairwise disjoint family of nonempty open sets of $\mathcal{F}[X]$. Thus, we have $c(\mathcal{F}[X]) \geq \tau^+$ and $hL(X) \leq c(\mathcal{F}[X])$.

Let $Y$ be a subspace of $X$. Consider the system $\mathcal{B}$ of all sets of the form $[A, U]$ where $A \in \mathcal{F}[Y]$ and $U$ is an open set of $X$ containing $A$. Let $\mathcal{B}$ be a pairwise disjoint maximal family of sets in $\mathcal{B}$. Then $D = \cup \{A : [A, U] \in \mathcal{B}\}$ is dense in $Y$. Because, assume there is $x \in Y \setminus \text{cl}_X D$. Then, there is an open nbd $V$ of $x$ in $X$ such that $V \cap D = \emptyset$. Since $\{\{x, V\} \cap [A, U] \subseteq \emptyset$ for each $[A, U] \in \mathcal{B}$, $\mathcal{B} \cup \{\{x, V\}\}$ is a pairwise disjoint family of sets of $\mathcal{B}$. This contradicts the maximality of $\mathcal{B}$. Since $|D| \leq |\mathcal{B}| \leq c(\mathcal{F}[X])$, we have $d(Y) \leq c(\mathcal{F}[X])$. Thus, $hd(X) \leq c(\mathcal{F}[X])$ is proved. This completes the proof of Theorem 2.

Hajnal and Juhász studied conditions when $\mathcal{F}[X]$ has countable cellularity. Readers, who take interest in this case, refer to [HJ].

Since $\mathcal{F}[X]$ is regular [vD], we have the following:

**Corollary 3.** In the following, (1)–(5) are equivalent, and each of them implies (6).

1. $X$ is countable.
2. $\mathcal{F}[X]$ is countable.
(3) $\mathbb{T}[X]$ is a cosmic space.
(4) $\mathbb{T}[X]$ is (hereditarily) separable.
(5) $\mathbb{T}[X]$ is (hereditarily) Lindelöf.
(6) $\mathbb{T}[X]$ is paracompact.

**Lemma 4.** If $X$ is a paracompact Hausdorff space, then $L(X) \leq c(X)$.

**Proof.** Since each open covering of $X$ has a $\sigma$-discrete open refinement, the lemma is clear.

**Corollary 5.** If $\nw(X) < |X|$ holds, then $\mathbb{T}[X]$ is not paracompact. In particular, if $X$ is an uncountable separable metric space, then $\mathbb{T}[X]$ is not metrizable.

The following corollaries are also clear.

**Corollary 6.** If $\mathbb{T}[X]$ has $G_δ$-diagonal, then the pseudo character of $X$ is $ω$.

**Corollary 7.** If $\mathbb{T}[X]$ has countable cellularity, then $X$ is both hereditarily separable and hereditarily Lindelöf.

**Theorem 8.** Let $\langle ϕ \rangle$ be a cardinal function listed above except $psw$, $ψ_δ$ and $Ψ$. Then, we have $ϕ(X) ≤ ϕ(\mathbb{T}[X])$ for each space $X$.

**Proof.** $t(X) \leq t(\mathbb{T}[X])$ follows from Proposition 1. Proofs for other cardinal functions follow from Theorem 2.

**Example 1.** By (4) in Theorem 2, we have always $psw(\mathbb{T}[X]) ≤ psw(X)$. There is a compact Hausdorff GO-space $X$ for which $psw(\mathbb{T}[X]) < psw(X)$ and $Ψ(\mathbb{T}[X]) < Ψ(X)$ hold. For example, let $X$ be the lexicographic ordered square $[E, 3.12.3]$. Then $psw(\mathbb{T}[X]) = ψ_δ(X) = Ψ(X) = c > ω = ψ(X)$. Therefore, we have $psw(\mathbb{T}[X]) = ψ_δ(\mathbb{T}[X]) = Ψ(\mathbb{T}[X]) = c$.

**Example 2.** There is a compact Hausdorff GO-space $X$ for which $hd(X)hL(X) < c(\mathbb{T}[X]) = nw(X)$ holds. In fact, let $X$ be the two arrows space $[E, 3.10.C]$. That is, $X = \{(x, 0): 0 < x < 1\} \cup \{(x, 1): 0 < x < 1\}$ considered as a subspace of the lexicographic ordered square. Then, $hd(X) = hL(X) = ω < c = nw(X)$. Let us take a point $P(x) = (x, 0)$ in $X$. Since $U(x) = \{(x, 0), (0, 1)\} \cup \{(y, i): 0 < y < x, i = 0, 1\}$ is an open nbd of $P(x)$, $\{|(P(x), U(x)): 0 < x < 1\}$ is a disjoint family of open sets of $\mathbb{T}[X]$. Therefore, we have $c(\mathbb{T}[X]) = c$.

**Example 3.** There is a $T_1$-space $X$ which has a finite subset $A$ such that $X(A, \mathbb{T}[X]) < \max\{x(x, X): x \in A\}$ holds. Let $X$ be the set of all lattice points $(i, j)$ of positive integers with two ideal points $p$ and $q$. The topology of $X$ is defined by declaring each lattice point to be open, and by taking as open nbd of $p$ sets of the form $X \setminus (B \cup \{q\})$ where $B$ is any set of lattice points with at most finitely many points on each row, and as open nbd of $q$ sets of the form $X \setminus (C \cup \{p\})$ where $C$ is any set of lattice points selected from at most finitely many rows. Then $X$ is a compact $T_1$-space and $x(p, X) > ω$ holds [SS, 99]. Let $A = \{p, q\}$. If $U$ is a nbd of $A$, then $|X \setminus U| < ω$. Therefore, $B = \{[A, U]: A \subset U \subset X, |X \setminus U| < ω\}$ is a nbd base of $A$ in $\mathbb{T}[X]$. Since $|B| = ω$, we have $x(A, \mathbb{T}[X]) < x(p, X)$.

Concerning theorems and examples, we ask the following.

**Question 1.** Let $ϕ ∈ \{Ψ, ψ_δ\}$. Does $ϕ(\mathbb{T}[X]) ≤ ϕ(X)$ hold for each space $X$?
Question 2. Determine exactly $t$, $\Psi$ and $\psi_A$ on $\mathcal{F}(X)$ in terms of those on $X$. Lutzer asked this question for the cellularity $c$ [L].

References