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CARDINAL FUNCTIONS ON PIXLEY-ROY HYPERSPACES

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ABSTRACT. Some cardinal functions on the Pixley-Roy hyperspace $\mathfrak{F}[X]$ of a space X are determined by those on X, and conditions of X by cardinal functions from which $\mathfrak{F}[X]$ has or has not some properties, e.g. cosmic, paracompact, ccc, etc. are given.

All spaces considered in this paper will be assumed to be infinite T_1-spaces. Cardinals are the initial ordinals, τ will always denote an infinite cardinal and ω is the smallest infinite cardinal. The cardinality of a set X is denoted by $|X|$ and $\mathfrak{c}$ is the cardinality of the continuum. The successor cardinal of τ is denoted by τ^+.

The *Pixley-Roy hyperspace* $\mathfrak{F}[X]$ of a space X has as its underlying set the collection of all finite nonempty subsets of X. If $A \in \mathfrak{F}[X]$, then the basic nbds (= neighbourhoods) of A are $[A, U] = \{S \in \mathfrak{F}[X]: A \subset S \subset U\}$, where U is an open set of X containing A.

We refer to [**J**] for the following cardinal functions—w (*weight*), π (π-*weight*), nw (*net weight*), d (*density*), c (*cellularity*), L (*Lindelöf-degree*), p (= e (*extent*) in [**E**]), χ (*character*), ψ (*pseudo character*), Ψ ($\Psi(X) = \omega$ iff X is perfect), ψ_Δ (*diagonal degree*), $\pi\chi$ (π-*character*), t (*tightness*) and *psw* (*point separating weight*). However, following Engelking [**E**], the *hereditary density* and the *hereditary Lindelöf-degree* are denoted by hd and hL, respectively.

In the following, Theorem 2 is the main result of this paper. Some results in it were given by van Douwen [**vD**], Lutzer [**L**] and Przymusiński [**P**]. Though Proposition 1 is an auxiliary one for Theorem 2, it is interesting in itself.

PROPOSITION 1. *If $A \in \mathfrak{F}[X]$, then the following hold*:

(1) $\psi(A, \mathfrak{F}[X]) = \max\{\psi(x, X): x \in A\}$.

(2) $\pi\chi(A, \mathfrak{F}[X]) = \chi(A, \mathfrak{F}[X]) \leqslant \max\{\chi(x, X): x \in A\}$.

If $x \in X$, then the following hold:

(3) $\chi(\{x\}, \mathfrak{F}[X]) = \chi(x, X)$.

(4) $t(x, X) \leqslant t(\{x\}, \mathfrak{F}[X])$.

If X is Hausdorff and $A \in \mathfrak{F}[X]$, then the following holds:

(5) $\chi(A, \mathfrak{F}[X]) = \max\{\chi(x, X): x \in A\}$.

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PROOF. (1) Let $A = \{x_1, \ldots, x_n\} \in \mathfrak{F}[X]$. For each $i \in \{1, \ldots, n\}$, there is a local ψ-base $\mathfrak{U}_i$ of x_i in X such that $|\mathfrak{U}_i| \leqslant \psi(x_i, X)$. Let

$$\mathfrak{U} = \{\cup \{U_i : i = 1, \ldots, n\} : U_i \in \mathfrak{U}_i\} \quad \text{and} \quad \mathfrak{B} = \{[A, U] : U \in \mathfrak{U}\}.$$

Then $\mathfrak{B}$ is a local ψ-base of A in $\mathfrak{F}[X]$ and $|\mathfrak{B}| \leqslant \max\{\psi(x_i, X) : i = 1, \ldots, n\}$. Thus, we have $\psi(A, \mathfrak{F}[X]) \leqslant \max\{\psi(x, X) : x \in A\}$. Conversely, let $\mathfrak{B}$ be a local ψ-base of A in $\mathfrak{F}[X]$ and $x \in A$. Then $\mathfrak{U} = \{B \setminus (A \setminus \{x\}) : [A, B] \in \mathfrak{B}\}$ is a local ψ-base of x in X. Therefore, we have $\psi(x, X) \leqslant \psi(A, \mathfrak{F}[X])$ and $\max\{\psi(x, X) : x \in A\} \leqslant \psi(A, \mathfrak{F}[X])$.

(2) Let $\mathfrak{B}$ be a local π-base of A in $\mathfrak{F}[X]$. For each $B \in \mathfrak{B}$, we can take an element $A_B \in B$ and an open nbd U_B of A_B in X such that $[A_B, U_B] \subset B$. Then, $\mathfrak{U} = \{[A, U_B] : B \in \mathfrak{B}, A \subset U_B\}$ is a nbd base of A in $\mathfrak{F}[X]$. For, let V be an open nbd of A in X. Then, there is $B \in \mathfrak{B}$ such that $B \subset [A, V]$. Therefore, $[A_B, U_B] \subset B \subset [A, V]$, i.e. $A \subset A_B \subset U_B \subset V$. Thus, $[A, U_B] \subset [A, V]$ and $\mathfrak{U}$ is a nbd base of A in $\mathfrak{F}[X]$. Since $|\mathfrak{U}| \leqslant |\mathfrak{B}|$, we have $\chi(A, \mathfrak{F}[X]) \leqslant \pi\chi(A, \mathfrak{F}[X])$. $\pi\chi(A, \mathfrak{F}[X]) \leqslant \chi(A, \mathfrak{F}[X])$ holds always and the other part is proved similar to (1).

(3) The proof is clear.

(4) Let $H \subset X$, $x \in X \setminus H$ and $x \in \mathrm{cl}_X H$. Let us put $\mathcal{H} = \mathfrak{F}[H \cup \{x\}] \setminus \{\{x\}\}$. Then $\{x\} \in \mathrm{cl}_{\mathfrak{F}[X]} \mathcal{H}$. Therefore, there is a subset $\mathfrak{M}$ of $\mathcal{H}$ such that $|\mathfrak{M}| \leqslant t(\{x\}, \mathfrak{F}[X])$ and $\{x\} \in \mathrm{cl}_{\mathfrak{F}[X]} \mathfrak{M}$. Let $M = (\cup \mathfrak{M}) \setminus \{x\}$. Then $x \in \mathrm{cl}_X M$. For, let V be a nbd of x in X. Since $\{x\} \in \mathrm{cl}_{\mathfrak{F}[X]} \mathfrak{M}$, there is an element $A \in [\{x\}, V] \cap \mathfrak{M}$. Then, $x \in A \subset V$ and $A \in \mathfrak{M}$. Since $A \neq \{x\}$, we have $V \cap M \neq \varnothing$ and $x \in \mathrm{cl}_X M$. Since $M \subset H$ and $|M| \leqslant |\mathfrak{M}| \leqslant t(\{x\}, \mathfrak{F}[X])$, we have $t(x, X) \leqslant t(\{x\}, \mathfrak{F}[X])$.

(5) Let us assume that there is $x \in A$ with $\chi(A, \mathfrak{F}[X]) < \chi(x, X)$. Let $\tau = \chi(A, \mathfrak{F}[X])$ and $\mathfrak{U}$ a nbd base of A in $\mathfrak{F}[X]$ with $|\mathfrak{U}| \leqslant \tau$. We can assume $\mathfrak{U} = \{[A, U_\alpha] : \alpha < \tau\}$, where U_α is an open set of X containing A. Since X is Hausdorff, there is an open nbd U of x with $(A \setminus \{x\}) \cap \mathrm{cl}_X U = \varnothing$. Since $\chi(x, X) > \tau$, there is an open nbd V of x with $U \cap U_\alpha \not\subset V$ for each $\alpha < \tau$. Let $W = V \cup (X \setminus \mathrm{cl}_X U)$. Then $[A, W]$ is an open nbd of A in $\mathfrak{F}[X]$. Since $U_\alpha \not\subset W$, $[A, U_\alpha] \not\subset [A, W]$ for each $\alpha < \tau$. Therefore, $\mathfrak{U}$ is not a nbd base of A, a contradiction. Thus, we have $\chi(x, X) \leqslant \chi(A, \mathfrak{F}[X])$ for each $x \in A$. This completes the proof of Proposition 1.

THEOREM 2. *For each space X, the following hold*:

(1) $|\mathfrak{F}[X]| = nw(\mathfrak{F}[X]) = d(\mathfrak{F}[X]) = L(\mathfrak{F}[X]) = p(\mathfrak{F}[X]) = |X|$.

(2) $\chi(\mathfrak{F}[X]) = \pi\chi(\mathfrak{F}[X]) = \chi(X)$.

(3) $w(\mathfrak{F}[X]) = \pi(\mathfrak{F}[X]) = \chi(X)|X|$.

(4) $psw(\mathfrak{F}[X]) = \psi(\mathfrak{F}[X]) = \psi(X)$.

(5) $\psi(X) = \omega$ *iff* $\Psi(\mathfrak{F}[X]) = \omega$ *iff* $\psi_\Delta(\mathfrak{F}[X]) = \omega$.

(6) $hd(X) hL(X) \leqslant c(\mathfrak{F}[X]) \leqslant nw(X)$.

PROOF. (1) Since $\{\{x\} : x \in X\}$ is a closed discrete subspace of $\mathfrak{F}[X]$, $|X| \leqslant p(\mathfrak{F}[X])$. $|\mathfrak{F}[X]| = d(\mathfrak{F}[X]) = |X|$ appears in **[vD]**. Therefore, (1) is clear.

(2) By Proposition 1, this is trivial.

(3) From

$$\chi(X) = \pi\chi(\mathfrak{F}[X]) \leqslant \pi(\mathfrak{F}[X]) \quad \text{and} \quad |X| = d(\mathfrak{F}[X]) \leqslant \pi(\mathfrak{F}[X]),$$

we have $\chi(X)|X| \leqslant \pi(\mathfrak{F}[X]) \leqslant w(\mathfrak{F}[X])$. On the other hand,

$$w(\mathfrak{F}[X]) \leqslant \chi(\mathfrak{F}[X])|\mathfrak{F}[X]|$$

is always true. Therefore, by (1) and (2), (3) is proved.

(4) Since $\psi(\mathfrak{F}[X]) \leqslant psw(\mathfrak{F}[X])$ is well known, we will prove $psw(\mathfrak{F}[X]) \leqslant \psi(X)$ only. For each $A \in \mathfrak{F}[X]$, there is a local ψ-base $\mathfrak{U}(A)$ of A with $|\mathfrak{U}(A)| \leqslant \psi(\mathfrak{F}[X]) = \psi(X)$. Then, $\mathfrak{B} = \{[A, U]: A \in \mathfrak{F}[X], U \in \mathfrak{U}(A)\}$ is a pseudo base of $\mathfrak{F}[X]$. Let $S \in \mathfrak{F}[X]$. $S \in [A, U]$ iff $A \subset S \subset U$. Since $|S| < \omega$, $\{A: \varnothing \neq A \subset S\}$ is finite. Therefore, $\operatorname{ord}(S, \mathfrak{B}) = |\{[A, U] \in \mathfrak{B}: S \in [A, U]\}| \leqslant \psi(X)$. Thus we have $\operatorname{ord}(\mathfrak{B}) = \sup\{\operatorname{ord}(S, \mathfrak{B}): S \in \mathfrak{F}[X]\} \leqslant \psi(X)$. Therefore, $psw(\mathfrak{F}[X]) \leqslant \psi(X)$.

(5) $\psi(X) = \omega$ iff $\Psi(\mathfrak{F}[X]) = \omega$ appears in **[L]** and **[P]**, and $\psi(\mathfrak{F}[X]) \leqslant \psi_\Delta(\mathfrak{F}[X])$ is well known. Therefore, only $\psi_\Delta(\mathfrak{F}[X]) \leqslant \psi(\mathfrak{F}[X]) = \omega$ needs proof. Let $\psi(\mathfrak{F}[X]) = \omega$ and $\mathfrak{U}(A) = \{U_n(A): n < \omega\}$ be a decreasing countable local ψ-base of each $A \in \mathfrak{F}[X]$ in $\mathfrak{F}[X]$. We can assume that $U_n(A)$ is of the form $[A, V]$, where V is an open nbd of A in X. Since $B_n = \bigcup\{U_n(A) \times U_n(A): A \in \mathfrak{F}[X]\}$ is an open nbd of the diagonal $\Delta(\mathfrak{F}[X]) = \{(A, A): A \in \mathfrak{F}[X]\}$ in $\mathfrak{F}[X]^2 = \mathfrak{F}[X] \times \mathfrak{F}[X]$, $\mathfrak{B} = \{B_n: n < \omega\}$ is a countable local ψ-base of $\Delta(\mathfrak{F}[X])$ in $\mathfrak{F}[X]^2$. For, let $(S, T) \in \mathfrak{F}[X]^2 \setminus \Delta(\mathfrak{F}[X])$. We may assume that there is an element $t \in T \setminus S$. For each nonempty $A \subset S$, there exists $n(A) < \omega$ such that $\{t\} \notin U_{n(A)}(A)$. If we put $n_0 = \max\{n(A): \varnothing \neq A \subset S\}$, then $T \notin \bigcup\{U_n(A): \varnothing \neq A \subset S\}$ for each $n \geqslant n_0$. Therefore, $(S, T) \notin B_{n_0}$ and $\mathfrak{B}$ is a local ψ-base of $\Delta(\mathfrak{F}[X])$ in $\mathfrak{F}[X]^2$. Therefore, we have $\psi_\Delta(\mathfrak{F}[X]) = \omega$.

(6) Since $c(\mathfrak{F}[X]) \leqslant nw(X)$ appears in **[L]**, it suffices to show $hd(X)hL(X) \leqslant c(\mathfrak{F}[X])$.

Let us assume $hL(X) > \tau$. Then, there are a subspace Y of X and an open covering $\mathcal{G}$ of Y such that $Y \not\subset \bigcup \mathcal{G}'$ for each subfamily $\mathcal{G}'$ of $\mathcal{G}$ with $|\mathcal{G}'| \leqslant \tau$. We can assume that each element of $\mathcal{G}$ is open in X. By transfinite induction, we can easily construct $\{x_\alpha: \alpha < \tau^+\} \subset Y$ and $\{G_\alpha: \alpha < \tau^+\} \subset \mathcal{G}$ with $x_\alpha \in G_\alpha \setminus (\bigcup\{G_\beta: \beta < \alpha\})$ for each $\alpha < \tau^+$. If $\beta < \alpha < \tau^+$, since $x_\alpha \notin G_\beta$, we have $[\{x_\beta\}, G_\beta] \cap [\{x_\alpha\}, G_\alpha] = \varnothing$. Thus, $\{[\{x_\alpha\}, G_\alpha]: \alpha < \tau^+\}$ is a pairwise disjoint family of nonempty open sets of $\mathfrak{F}[X]$. Thus, we have $c(\mathfrak{F}[X]) \geqslant \tau^+$ and $hL(X) \leqslant c(\mathfrak{F}[X])$.

Let Y be a subspace of X. Consider the system $\mathfrak{U}$ of all sets of the form $[A, U]$ where $A \in \mathfrak{F}[Y]$ and U is an open set of X containing A. Let $\mathfrak{B}$ be a pairwise disjoint maximal family of sets in $\mathfrak{U}$. Then $D = \bigcup\{A: [A, U] \in \mathfrak{B}\}$ is dense in Y. Because, assume there is $x \in Y \setminus \operatorname{cl}_X D$. Then, there is an open nbd V of x in X such that $V \cap D = \varnothing$. Since $[\{x\}, V] \cap [A, U] = \varnothing$ for each $[A, U] \in \mathfrak{B}$, $\mathfrak{B} \cup \{[\{x\}, V]\}$ is a pairwise disjoint family of sets in $\mathfrak{U}$. This contradicts the maximality of $\mathfrak{B}$. Since $|D| \leqslant |\mathfrak{B}| \leqslant c(\mathfrak{F}[X])$, we have $d(Y) \leqslant c(\mathfrak{F}[X])$. Thus, $hd(X) \leqslant c(\mathfrak{F}[X])$ is proved. This completes the proof of Theorem 2.

Hajnal and Juhász studied conditions when $\mathfrak{F}[X]$ has countable cellularity. Readers, who take interest in this case, refer to **[HJ]**.

Since $\mathfrak{F}[X]$ is regular **[vD]**, we have the following:

COROLLARY 3. *In the following*, (1)–(5) *are equivalent, and each of them implies* (6).

(1) *X is countable.*

(2) *$\mathfrak{F}[X]$ is countable.*

(3) $\mathfrak{F}[X]$ *is a cosmic space.*
(4) $\mathfrak{F}[X]$ *is (hereditarily) separable.*
(5) $\mathfrak{F}[X]$ *is (hereditarily) Lindelöf.*
(6) $\mathfrak{F}[X]$ *is paracompact.*

LEMMA 4. *If X is a paracompact Hausdorff space, then $L(X) \leqslant c(X)$.*

PROOF. Since each open covering of X has a σ-discrete open refinement, the lemma is clear.

COROLLARY 5. *If $nw(X) <|X|$ holds, then $\mathfrak{F}[X]$ is not paracompact. In particular, if X is an uncountable separable metric space, then $\mathfrak{F}[X]$ is not metrizable.*

The following corollaries are also clear.

COROLLARY 6. *If $\mathfrak{F}[X]$ has G_δ-diagonal, then the pseudo character of X is ω.*

COROLLARY 7. *If $\mathfrak{F}[X]$ has countable cellularity, then X is both hereditarily separable and hereditarily Lindelöf.*

THEOREM 8. *Let ϕ be a cardinal function listed above except psw, ψ_Δ and Ψ. Then, we have $\phi(X) \leqslant \phi(\mathfrak{F}[X])$ for each space X.*

PROOF. $t(X) \leqslant t(\mathfrak{F}[X])$ follows from Proposition 1. Proofs for other cardinal functions follow from Theorem 2.

EXAMPLE 1. By (4) in Theorem 2, we have always $psw(\mathfrak{F}[X]) \leqslant psw(X)$. There is a compact Hausdorff GO-space X for which $psw(\mathfrak{F}[X]) < psw(X)$, $\psi_\Delta(\mathfrak{F}[X]) < \psi_\Delta(X)$ and $\Psi(\mathfrak{F}[X]) < \Psi(X)$ hold. For example, let X be the lexicographic ordered square [**E**, 3.12.3]. Then $psw(X) = \psi_\Delta(X) = \Psi(X) = \mathfrak{c} > \omega = \psi(X)$. Therefore, we have $psw(\mathfrak{F}[X]) = \psi_\Delta(\mathfrak{F}[X]) = \Psi(\mathfrak{F}[X]) = \omega$ by Theorem 2.

EXAMPLE 2. There is a compact Hausdorff GO-space X for which $hd(X)hL(X) < c(\mathfrak{F}[X]) = nw(X)$ holds. In fact, let X be the two arrows space [**E**, 3.10.C]. That is, $X = \{(x,0)\colon\ 0 < x \leqslant 1\} \cup \{(x,1)\colon\ 0 \leqslant x < 1\}$ considered as a subspace of the lexicographic ordered square. Then, $hd(X) = hL(X) = \omega < \mathfrak{c} = nw(X)$. Let us take a point $P(x) = (x,0)$ in X. Since $U(x) = \{(x,0),\ (0,1)\} \cup \{(y,i)\colon\ 0 < y < x,\ i = 0,1\}$ is an open nbd of $P(x)$, $\{[\{P(x)\}, U(x)]\colon 0 < x \leqslant 1\}$ is a disjoint family of open sets of $\mathfrak{F}[X]$. Therefore, we have $c(\mathfrak{F}[X]) = \mathfrak{c}$.

EXAMPLE 3. There is a T_1-space X which has a finite subset A such that $\chi(A, \mathfrak{F}[X]) < \max\{\chi(x, X)\colon\ x \in A\}$ holds. Let X be the set of all lattice points (i, j) of positive integers with two ideal points p and q. The topology of X is defined by declaring each lattice point to be open, and by taking as open nbds of p sets of the form $X\backslash(B \cup \{q\})$ where B is any set of lattice points with at most finitely many points on each row, and as open nbds of q sets of the form $X\backslash(C \cup \{p\})$ where C is any set of lattice points selected from at most finitely many rows. Then X is a compact T_1-space and $\chi(p, X) > \omega$ holds [**SS**, 99]. Let $A = \{p, q\}$. If U is a nbd of A, then $|X\backslash U| < \omega$. Therefore, $\mathfrak{B} = \{[A, U]\colon A \subset U \subset X,\ |X\backslash U| < \omega\}$ is a nbd base of A in $\mathfrak{F}[X]$. Since $|\mathfrak{B}| = \omega$, we have $\chi(A, \mathfrak{F}[X]) < \chi(p, X)$.

Concerning theorems and examples, we ask the following.

Question 1. Let $\phi \in \{\Psi, \psi_\Delta\}$. Does $\phi(\mathfrak{F}[X]) \leqslant \phi(X)$ hold for each space X?

Question 2. Determine exactly t, Ψ and ψ_Δ on $\mathfrak{F}[X]$ in terms of those on X. Lutzer asked this question for the cellularity c [**L**].

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