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CH AND OPEN SUBSPACES OF F-SPACES

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ABSTRACT. N. J. Fine and L. Gillman showed that, if one assumes CH, each open subset of an F-space of weight c is an F-space. In this note it is shown that this fact is equivalent to CH.

0. Introduction. A Tychonoff space is an *F-space* if disjoint cozero subsets are contained in disjoint zero sets. Many interesting results about F-spaces of weight c have been shown to be consequences of the continuum hypothesis, CH; see for example [**Pa, L, FG, W1**, and **W2**]. Unfortunately most of these have also been shown to be equivalent to CH. For example, in [**vDvM1**] it is shown that Parovičenko's characterization of ω^* is equivalent to CH; in [**vD1**] it is shown that Woods' result, that countably compact normal F-spaces with only c continuous real-valued functions are compact, is equivalent to CH, and it has been shown that others are not theorems of ZFC [**vDvM2** and **vDvM3**]. In this note it is shown that Fine and Gillman's result, each open subset of an F-space of weight c is an F-space, is also equivalent to CH. It is also true, under CH, that each locally compact subspace with weight c of an F-space is again an F-space. We show that, under $\neg$CH, every infinite compact F-space contains a locally compact subspace of weight c which is not an F-space.

Our notation and terminology follows that of the Gillman and Jerison text [**GJ**]. We list some necessary facts about F-spaces.

0.1. PROPOSITION. (i) *X is an F-space iff each cozero subset of X is C^*-embedded.*
(ii) *X is an F-space iff βX is an F-space.*
(iii) *A C^*-embedded subspace of an F-space is an F-space.*
(iv) *Lindelöf subspaces of F-spaces are C^*-embedded.* □

We will be using the absolute $E(X)$ of a compact space X. There is a canonical continuous map k_X which maps $E(X)$, the space of maximal regular open ultrafilters, to X defined by $k_X(u) \in \bigcap\{\text{cl}_X U\colon U \in u\}$. For each regular open subset U of X, $\overline{k^{\leftarrow}(U)}$ is a clopen subset of $E(X)$ and these form a base for $E(X)$. Also if g is a homeomorphism of X then there is a unique homeomorphism h of $E(X)$ such that $k_X \circ h = g \circ k_X$. The reader should consult the survey [**W3**] for these and other facts about the absolute.

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1. Assuming CH. In this section we shall list and prove the relevant results from [**FG**] for completeness. The first result does not require CH.

1.1. THEOREM [**FG**]. *Each union of ω_1 cozero subsets of an F-space is again an F-space.*

PROOF. Let X be an F-space and $U = \bigcup\{C_\alpha\colon \alpha < \omega_1\}$ with each C_α a cozero subset of X. Let g be a bounded real-valued continuous function defined on a cozero subset C of U. We shall show that g extends continuously to $\hat{g}$ on U. Recursively define bounded real-valued continuous functions g_α on $\bigcup_{\beta<\alpha} C_\beta$ such that for $\beta < \alpha$, $g_\beta \subset g_\alpha$ and g_α restricted to $C \cap \bigcup_{\beta<\alpha} C_\beta$ is equal to g. This recursion can be carried out since, for each $\alpha < \omega_1$, $\bigcup_{\beta<\alpha} C_\beta \cup (C \cap C_\alpha)$ is a cozero subset of $\bigcup_{\beta\leqslant\alpha} C_\beta$ and therefore is C^*-embedded in $\bigcup_{\beta\leqslant\alpha} C_\beta$. Therefore $\hat{g} = \bigcup_{\alpha<\omega_1} g_\alpha$ is the desired extension of g. □

1.2. THEOREM (CH). *Each open subset of an F-space of weight c is an F-space.* □

1.3. THEOREM (CH). *Each locally compact space of weight c which is a subspace of an F-space is itself an F-space.*

PROOF. Let X be an F-space and let Y be a locally compact subspace which has weight c. By Proposition 0.1, $K = \mathrm{cl}_{\beta X} Y$ is an F-space. Since Y is locally compact and dense in K, Y is open in K. Clearly Y can be expressed as a union of $c = \omega_1$ cozero subsets of K. By Theorem 1.1 Y is an F-space. □

2. The example. In this section we shall construct an example of an F-space K with weight $\omega_2 \cdot c$ which has an open subspace U which is not an F-space. We shall need the following much weakened version of a result of Negrepontis. A space X is a *P-space* if each countable intersection of open subsets of X is open.

2.1. PROPOSITION. *If X is a P-space, then $X \times \omega$ is an F-space and $X \times \omega$ is C^*-embedded in $X \times \beta\omega$.*

PROOF. In fact, $X \times \omega$ is a P-space since it is the product of two P-spaces. It easily follows from the fact that ω is countable (or Lindelöf) that the projection map from $X \times \omega$ to X is z-closed [**Wa**]. Therefore $X \times \omega$ is C^*-embedded in $X \times \beta\omega$ [**Wa**, p. 193]. □

Let X be the set $\omega_2 + 1$ endowed with the G_δ-topology of the ordinal space $\omega_2 + 1$. Clearly X is a P-space and $L = \{\lambda \leqslant \omega_2\colon \lambda \text{ has uncountable cofinality}\}$ is the set of nonisolated points of X. It is not difficult to show that $|C^*(X)| = \omega_2 \cdot c$ and that X is Lindelöf (see 9L in [**GJ**]). Let $K = \beta(X \times \omega^*)$.

2.2. *Fact*. K is a compact F-space with weight $\omega_2 \cdot c$.

PROOF. Since $|C^*(X \times \omega)| = \omega_2 \cdot c$, by Proposition 2.1 $|C^*(X \times \beta\omega)| = \omega_2 \cdot c$. Hence the weight of $\beta(X \times \beta\omega)$ is $\omega_2 \cdot c$. Now since X, and therefore, $X \times \omega^*$ are Lindelöf, $X \times \omega^*$ is C^*-embedded in $\beta(X \times \omega)$ by Proposition 0.1(iv). It follows that K is an F-space and has weight $\omega_2 \cdot c$. □

Now let C be a nonclopen cozero subset of ω^*. Define $T = \mathrm{cl}_K(L \times C)$ and $U = K - T$.

2.3. *Fact*. U is not an F-space.

PROOF. We shall exhibit disjoint cozero subsets C_0 and C_1 of U which are not contained in disjoint zero sets of U. Let E_0 and E_1 be disjoint subsets of $X - L$ such that $L \subset \overline{E}_0 \cap \overline{E}_1$, e.g. E_0 is the even isolated ordinals and E_1 is the odd ordinals. Let W_n, for $n \in \omega$, be clopen in ω^* such that $\bigcup W_n = C$. Define $V_n(i) = \mathrm{cl}_U(E_i \times W_n)$ for $i = 0$ and $i = 1$. We show that each $V_n(i)$ is open. Suppose that $u \in \mathrm{cl}_U V_n(i)$ for $i = 0$ or 1.

Therefore $\mathrm{cl}_K(X \times W_n)$ is a clopen neighborhood of u in K. Let W be a clopen K-neighborhood of u contained in $\mathrm{cl}_K(X \times W_n)$ such that $W \cap (L \times C) = \varnothing$. Since W is compact there is a countable set I contained in $X \backslash L$ so that $W \subset \mathrm{cl}_K(I \times W_n)$. Now $(I \cap E_i) \times W_n$ is clopen in $X \times \omega^*$ and so $\mathrm{cl}_K((I \cap E_i) \times W_n)$ is a neighborhood of u which is contained in $V_n(i)$. Therefore $V_n(0)$ and $V_n(1)$ are clopen subsets of U for each $n \in \omega$. Let $C_i = \bigcup_n V_n(i)$ for $i = 0, 1$, these are clearly cozero subsets of U. We must consider two cases to show that C_0 and C_1 are not contained in disjoint zero sets of U.

Case 1. $\mathrm{cl}_{\omega^*} C$ is *not* the intersection of ω_1 clopen subsets of ω^*. Let $X_1 = X \cap \{\alpha: \alpha \leqslant \omega_1\}$, $K_1 = \mathrm{cl}_K(X_1 \times \omega^*)$ and $U_1 = K_1 \cap U$. Since X_1 is clopen in X, K_1 is clopen in K and U_1 is clopen in U. We can show in this case that not even U_1 is an F-space. Note that the weight of K_1 is c but, of course, by our current assumption $c \geqslant \omega_2$.

Suppose that $(U_1 \cap C_i) \subset Z_i$ for disjoint zero sets Z_i, $i = 0, 1$. It follows that for each $\alpha \in E_i \cap X_1$ there is a clopen subset, A_α, of ω^* with $\{\alpha\} \times A_\alpha \subset Z_i$ and $C \subset A_\alpha$. By assumption, there is a point $y \in \bigcap \{A_\alpha \backslash \mathrm{cl}_{\omega^*} C: \alpha \in E_0 \cup E_1\}$. Therefore $(\omega_1, y) \in U_1$, and $(\omega_1, y) \in Z_0 \cap Z_1$ because $X_1 \times \omega^*$ has the product topology. This contradicts the fact that Z_0 and Z_1 are assumed to be disjoint.

Case 2. $\mathrm{cl}_{\omega^*} C = \bigcap \{B_\xi: \xi < \omega_1\}$ where each B_ξ is a clopen subset of ω^*. We may assume that $\{B_\xi: \xi < \omega_1\}$ is closed under finite intersections and is therefore a neighborhood base for $\mathrm{cl}_{\omega^*} C$. Note that in this case U_1 is an F-space because it can be expressed as an ω_1 union of clopen subsets of K_1.

Suppose that $C_i \subset Z_i$ for disjoint zero sets Z_0 and Z_1. Again it follows that for $\alpha \in E_i$ there is a clopen $A_\alpha \subset \omega^*$ such that $\{\alpha\} \times A_\alpha \subset Z_i$ and $C \subset A_\alpha$. For each $\alpha \in E_0 \cup E_1$ choose $\xi(\alpha) < \omega_1$ so that $B_{\xi(\alpha)} \subset A_\alpha$. Therefore, for $i = 0, 1$; there is a set $E_i' \subset E_i$ with cardinality ω_2 and a $\xi_i < \omega_1$ such that $\xi(\alpha) = \xi_i$ for each $\alpha \in E_i'$. Let $p \in (B_{\xi_0} \cap B_{\xi_1}) \backslash \overline{C}$. Then $(\omega_2, p) \in \overline{Z}_1 \cap \overline{Z}_2$ and $(\omega_2, p) \in U$, which contradicts the disjointness of Z_0 and Z_1. □

It is fairly well known that both Case 1 and Case 2 above are consistent with ¬CH. This is shown, for example, in **[H]**.

REMARK. The underlying idea behind the construction of U is that U is like a product of a P-space X with an F-space Y where Y has the property that there are some cozero subsets of Y with no 'X-limits'. However, any zero sets containing them do have 'X-limits'. This idea is made more precise and is employed in **[D]** to find a necessary condition on F-spaces in order that their product with any P-space be F-spaces.

3. Assuming ¬CH. In this section we will show the opposite implications of our equivalences of CH. We shall need a result of van Douwen's **[vD2]** which we shall show in a pair of lemmas.

3.1. LEMMA. *The P-space* $(\omega^*)_\delta$ *embeds in* $\beta\omega$.

PROOF. Let $p \in \omega^*$ be arbitrary and recall the definition: $\{p\text{-lim}(x_n: n \in \omega)\} = \bigcap\{\overline{(x_n: n \in U)}: U \in p\}$. It is well known that $\beta(\omega \times \omega^*)$ can be embedded in ω^*. For $q \in \omega^*$, define $f(q) = p\text{-lim}\{(n, q): n \in \omega\} \in \beta(\omega \times \omega^*)$. We shall show that $Q = \{f(q): q \in \omega^*\}$ is homeomorphic to $(\omega^*)_\delta$ and that f is a homeomorphism. Let $(B_n: n \in \omega)$ be clopen subsets of ω^*. Clearly $\overline{\bigcup(\{n\} \times \bigcup_{k \leqslant n} B_k: n \in \omega)}$ is a clopen subset of $\omega \times \omega^*$ so $G = \overline{\bigcup_n\{n\} \times \bigcup_{k \leqslant n} B_k}$ is a clopen subset of $\beta(\omega \times \omega^*)$. It is easy to check that $G \cap Q = f[\bigcap_n B_n]$; hence f is open. Conversely, let $U \subset Q$ be a neighborhood of $f(q)$ in Q. Choose W, a clopen subset of $\beta(\omega \times \omega^*)$, so that $f(q) \in W \cap Q \subset U$. Since $f(q) \in W$, $W' = \{n: (n, q) \in W\} \in p$. For $n \in W'$, choose B_n so that $\{n\} \times B_n = W \cap (\{n\} \times \omega^*)$. We observe that $f[\bigcap(B_n: n \in W')] \subset W$ which shows that f is continuous. □

3.2. LEMMA. *The space* $(2^c)_\delta$ *embeds in* $\beta\omega$.

PROOF. Since 2^c is separable, $E(2^c)$ embeds in ω^*. Therefore $(E(2^c))_\delta$ is a subspace of $(\omega^*)_\delta$ and from the fact that $(\omega^*)_\delta$ embeds in ω^* it follows that $(E(2^c))_\delta$ embeds in ω^*. Hence it suffices to show that $(2^c)_\delta$ embeds in $(E(2^c))_\delta$. Let k be the canonical map from $E(2^c)$ to 2^c. We shall think of $E(2^c)$ as the set of ultrafilters of regular open subsets of 2^c. We will also let $+$ be coordinatewise addition modulo 2 on 2^c and $\mathbf{0}$ be the identity. Choose $u_0 \in k^{\leftarrow}(\mathbf{0})$ arbitrarily. For each $x \in 2^c$ we have the homeomorphism g_x of 2^c defined by $g_x(y) = x + y$. There is a unique homeomorphism h_x of $E(2^c)$ such that $g_x \circ k = k \circ h_x$. Let $Y = \{h_x(u_0): x \in 2^c\} \subset (E(2^c))_\delta$. To show that Y is homeomorphic to $(2^c)_\delta$ via k it suffices to show that if A is clopen in $E(2^c)$ then $k(A \cap Y)$ is open in $(2^c)_\delta$. To this end let $U \subset 2^c$ be regular open. Since 2^c is ccc (there are no uncountable cellular families of open sets), there are countably many basic clopen sets A_n such that $\bigcup A_n \subset U$ is dense. Let F_n, $n \in \omega$, be finite subsets of c such that A_n is only restricted on the coordinates in F_n and let $F = \bigcup_n F_n$. Suppose that $x, y \in 2^c$ are such that $x|_F = y|_F$, we shall show that $k^{\leftarrow}(x) \in \overline{k^{\leftarrow}(U)}$ iff $k^{\leftarrow}(y) \in \overline{k^{\leftarrow}(U)}$. Indeed, $k^{\leftarrow}(x) \in \overline{k^{\leftarrow}(U)}$ iff $U \in h_x(u_0)$ iff $\operatorname{int} \overline{\bigcup_n A_n} \in h_x(u_0)$ iff $\operatorname{int} \overline{\bigcup (A_n + x)} \in u_0$ iff $\operatorname{int} \overline{\bigcup (A_n + x + y)} \in h_y(u_0)$ iff $\operatorname{int} \overline{\bigcup A_n} \in h_y(u_0)$ (since $x|_F = y|_F$) iff $k^{\leftarrow}(y) \in \overline{k^{\leftarrow}(U)}$. Now F is countable so 2^F is homeomorphic to 2^ω which is first countable. It follows that $x|_F \times 2^{c-F}$ is clopen in $(2^c)_\delta$ for each $x \in 2^c$. Finally,

$$k\left(\overline{k^{\leftarrow}(U)} \cap Y\right) = \bigcup_n A_n \cup \bigcup \left\{x|_F \times 2^{c-F}: x \in k\left(\overline{k^{\leftarrow}(U)}\right)\right\}$$

which is open by the above. □

3.3. COROLLARY ($\neg$CH). *The space K constructed in* §2 *embeds in* ω^*.

PROOF. Recall that $X = (\omega_2 + 1)_\delta$ and $K = \beta(X \times \omega^*)$. As in §2, it suffices to show that $X \times \omega$ can be C^*-embedded in ω^*. However since X is Lindelöf and ω^* is an F-space it suffices to show that X embeds in ω^*. Since we are assuming $c > \omega_1$, we can embed X in 2^c. Now X is a P-space, hence X embeds in $(2^c)_\delta$ and by 3.2 in ω^*. □

We can now state our main results.

3.4. THEOREM. *The following are equivalent*:

(i) *CH*,

(ii) *each open subset of a compact F-space of weight c is an F-space*,

(iii) *each open subset of an F-space of weight c is an F-space*,

(iv) *each locally compact subspace of an F-space of weight c is an F-space*.

PROOF. Theorem 1.2, Theorem 1.3 and the example in §2. □

3.5. THEOREM ($\neg$CH). *Every infinite compact F-space contains a locally compact subspace which is not an F-space and has weight c.*

PROOF. Every infinite compact F-space contains ω^* and therefore, by Corollary 3.3, contains K. □

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