MAPPING THEOREMS ON MESOCOMPACT SPACES

KUO-SHIH KAO AND LI-SHENG WU

Abstract. In this paper we prove two mapping theorems on mesocompact spaces:
(1) Perfect mappings preserve mesocompactness; (2) Closed mappings preserve
normal mesocompactness.

The main results of this paper are two mapping theorems on mesocompact spaces. Mesocompactness was defined in J. R. Boone [4] and studied by V. J. Mancuso [10] and J. R. Boone [4, 5]. Mancuso [10] intended to prove that perfect mappings preserve mesocompactness, but his proof was incorrect. J. R. Boone [5] noticed the error in Mancuso's proof but he gave a proof only for a special case (the domains of the mappings were assumed to be normal). Our Theorem 1 solves the Mancuso problem. Boone [6] studied k-quotient mappings and proved that k-quotient, closed mappings preserve normal mesocompactness. Our Theorem 2 improves the foregoing result by omitting the condition "k-quotient" in the statement.

In this paper, normal spaces are assumed to be $T_1$, and all mappings are
continuous and surjective. Let $\mathcal{U}$ be a collection of subsets of $X$, the union
$\bigcup \{U: U \in \mathcal{U}\}$ is denoted by $\mathcal{U}^*$. For any $B \subseteq X$, let $(\mathcal{U})_B = \{U \in \mathcal{U}: U \cap B \neq \emptyset\}$ and $(\mathcal{U})_x$ is replaced by $(\mathcal{U})_x$. For the meanings of concepts used without
definition in this paper, see [8 and 9].

Definition 1. A collection $\mathcal{U}$ of subsets of $X$ is called compact-finite, if for each
compact subset $K \subseteq X$, $(\mathcal{U})_K$ is finite.

Definition 2 [4]. A topological space $X$ is called mesocompact if every open cover
of the space has a compact-finite open refinement.

It is well known that
paracompact $\rightarrow$ mesocompact $\rightarrow$ metacompact

and none of the implications can be reversed.

Definition 3. Let $\mathcal{U}$ and $\mathcal{V}$ be two collections of subsets of $X$, we say that $\mathcal{U}$ is a
compactwise $W$-refinement of $\mathcal{V}$, if $\mathcal{U}^* = \mathcal{V}^*$ and for each compact subset $K \subseteq X$, the
collection $(\mathcal{U})_K$ is a partial refinement of some finite subcollection $\mathcal{V}'$ of $\mathcal{V}$.

In the proofs of the following lemmas and Proposition 1, we use the techniques
invented by Junnila [8 and 9].

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Lemma 1. Let $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a sequence of open covers of $X$ such that for each $n \in \mathbb{N}$, $\mathcal{U}_{n+1}$ is a compactwise $W$-refinement of $\mathcal{U}_n$. Then $\mathcal{U}_1$ has an open refinement $\mathcal{V} = \bigcup_{n=2}^{\infty} \mathcal{V}_n$ such that each $\mathcal{V}_n$ is a compact-finite collection.

Proof. Because each $\mathcal{U}_{n+1}$ is also a pointwise $W$-refinement of $\mathcal{U}_n$, by [8, Proposition 2.2], $\mathcal{U}_1$ has an open refinement $\mathcal{V} = \bigcup_{n=2}^{\infty} \mathcal{V}_n$, such that for each $n \in \mathbb{N}$ and $B \subset X$, if $(\mathcal{U}_{n+1})_B$ is a partial refinement of a subcollection $\mathcal{U}'$ of $\mathcal{U}_n$, then $|(\mathcal{V}_{n+1})_B| \leq |\mathcal{U}'|$. Now for each compact subset $K \subset X$, there exists a finite subcollection $\mathcal{U}'$ of $\mathcal{U}_n$ such that $(\mathcal{U}_{n+1})_K$ is a partial refinement of $\mathcal{U}'$, so $|(\mathcal{V}_{n+1})_K| \leq |\mathcal{U}'| < \infty$, that is, $\mathcal{V}_n$ is compact-finite for each $n \geq 2$.

Lemma 2. If an open cover of a topological space has a compact-finite semiopen refinement, then the cover has an open compactwise $W$-refinement.

Proof. The proof is similar to the proof of [9, Lemma 1.2].

Proposition 1. The following conditions are mutually equivalent for a topological space:

1. The space is mesocompact.
2. Every open cover of the space has a compact-finite semiopen refinement.
3. Every open cover of the space has an open compactwise $W$-refinement.
4. Every directed open cover of the space has a closure-preserving closed refinement which is refined by the collection consisting of all compact subsets.

Proof. (1) $\Rightarrow$ (2) is obvious. (2) $\Rightarrow$ (3) follows from Lemma 2.

(3) $\Rightarrow$ (1). By Lemma 1, every open cover $\mathcal{U}$ of $X$ has an open refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, each $\mathcal{V}_n$ is compact-finite. For each $n \in \mathbb{N}$, let $R_n = \bigcup_{k=1}^{n} \mathcal{V}_k$, then $\mathcal{R} = \{R_n; n \in \mathbb{N}\}$ is a directed open cover of $X$. Let $\mathcal{P}$ be an open compactwise $W$-refinement of $\mathcal{R}$. Let $F_0 = \emptyset$ and for each $n \in \mathbb{N}$, let $F_n = \{x \in X: \text{St}(x, \mathcal{P}) \subset R_n\}$, note that each set $F_n$ is closed. For each $n \in \mathbb{N}$, let $\mathcal{V}_n = \{W - F_n: W \in \mathcal{V}_n\}$, it is easily seen that $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ is an open refinement of $\mathcal{U}$ [8, Theorem 3.1, proof (ii) $\Rightarrow$ (i)].

Let $V \subset X$ be any compact subset. Because $\mathcal{R}$ is directed, there exists an integer $n(K)$ such that $\text{St}(K, \mathcal{P}) \subset R_{n(K)}$, that is $K \subset F_{n(K)}$. Then we have

$$|\mathcal{V}_n| \leq \sum_{n=1}^{\infty} |(\mathcal{V}_n)_K| = \sum_{n=1}^{n(K)} |(\mathcal{V}_n)_K| \leq \sum_{n=1}^{n(K)} |(\mathcal{U}_n)_K| < \infty.$$ 

Therefore $\mathcal{U}$ is compact-finite.

(1) $\Rightarrow$ (4). Let $\mathcal{U}$ be any directed open cover of $X$, then $\mathcal{U}$ has a compact-finite open refinement $\mathcal{V}$. For each $U \in \mathcal{U}$, let $F(U) = \{x \in X: \text{St}(x, \mathcal{V}) \subset U\}$, then $\mathcal{F} = \{F(U): U \in \mathcal{U}\}$ is a closure-preserving closed refinement of $\mathcal{U}$ [8, Lemma 2.3, proof (ii) $\Rightarrow$ (i)]. Suppose $K \subset X$ is a compact subset. Since $\mathcal{U}$ is directed, there exists some $U \in \mathcal{U}$, such that $\text{St}(K, \mathcal{V}) \subset U$, that is $K \subset F(U)$. Therefore $\mathcal{F}$ is refined by the collection consisting of all compact subsets.

(4) $\Rightarrow$ (3). By [8, Theorem 3.1], $X$ is metacompact. Suppose $\mathcal{U}$ is any open cover of $X$, then $\mathcal{U}$ has a point-finite open refinement $\mathcal{V}$. Let $\mathcal{Y}$ be the collection consisting of...
all finite unions of sets from $\mathcal{V}$. As a directed open cover of $X$, $\mathcal{V}$ has a closure-preserving closed refinement $\mathcal{F}$ which is refined by the collection consisting of all compact subsets. For each $x \in X$, let $W(x) = \bigcap (\mathcal{V}_x) - \bigcup \{ F \in \mathcal{F} : x \notin F \}$. $\mathcal{W} = \{ W(x) : x \in X \}$ is an open cover of $X$. For each $F \in \mathcal{F}$, let $\mathcal{V}_F$ be the finite subcollection of $\mathcal{V}$ such that $F \subseteq \bigcap \mathcal{V}_F \in \mathcal{V}$. Then $(\mathcal{W})_F$ is a partial refinement of $\mathcal{V}_F$.

Now for each compact subset $K \subset X$, there exists an $F(K) \in \mathcal{F}$ such that $K \subseteq F(K)$, so $(\mathcal{W})_K$ is a partial refinement of a finite subcollection $\mathcal{V}_{F(K)}$ of $\mathcal{V}$, that is, $\mathcal{W}$ is an open compactwise $W$-refinement of $\mathcal{V}$ as well as of $\mathcal{W}$.

**Definition 4 (Michael [12]).** A mapping $f: X \rightarrow Y$ is called compact-covering if, whenever $K$ is a compact set in $Y$, there exists a compact set $C$ in $X$ such that $f(C) = K$.

**Proposition 2.** The image of a mesocompact space under a closed and compact-covering mapping is mesocompact.

**Proof.** Let $\mathcal{V} = \{ V_b \}_{b \in B}$ be a directed open cover of $Y$, then $\mathcal{U} = \{ f^{-1}(V_b) \}_{b \in B}$ is a directed open cover of $X$. By Proposition 1(4), $\mathcal{U}$ has a closure-preserving closed refinement $\mathcal{F} = \{ F_a \}_{a \in A}$, which is refined by the collection consisting of all compact subsets of $X$. Since $f$ is closed and compact-covering, $\{ f(F_a) \}_{a \in A}$ is a closure-preserving closed refinement of $\mathcal{V}$, and is refined by the collection consisting of all compact subsets of $Y$. By Proposition 1(4), $Y$ is mesocompact.

Since every perfect mapping is compact-covering (e.g. [13]), we obtain the following theorem.

**Theorem 1.** The image of a mesocompact space under a perfect mapping is mesocompact.

**Definition 5 (Bacon [3]).** A topological space is called isocompact if every countably compact closed subset of $X$ is compact.

E. Michael [11] proved that if $X$ is a paracompact space and $f: X \rightarrow Y$ a closed mapping, then $f$ is also a compact-covering mapping. In his proof, the paracompactness is used only for turning a countably compact closed subset of a normal space to a compact one. We have the following lemma.

**Lemma 3.** A closed mapping $f$ from a normal isocompact space $X$ onto a space $Y$ is also a compact-covering mapping.

**Theorem 2.** The image of a normal mesocompact space under a closed mapping is normal mesocompact.

**Proof.** Let $f$ be a closed mapping from a normal mesocompact space $X$ onto a space $Y$. It is well known that $Y$ is normal. Because mesocompact space is isocompact (Arens-Dugundji [1]), so by Lemma 3, $f$ is also a compact-covering mapping. By Proposition 2, $Y$ is mesocompact.

**Remark.** Junnila [9] proved that the image of a paracompact space under a pseudo-open and compact mapping is metacompact in order to answer the question of Arhangel’skii [2] affirmatively. Junnila’s proof for the paracompact case can easily be modified to cover the mesocompact case. On the other hand, the image of a
metacompact space under an open and compact mapping is not necessarily metacompact (Chaber [7]).

References


Department of Mathematics, Kiangsu Teachers’ College, Suchow, Kiangsu, China