A SMOOTH SCISSORS CONGRUENCE PROBLEM

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Abstract. Classifying space techniques are used to solve a smooth version of the classical scissors congruence problem.

1. Introduction.

1.1 The classical problem [8]. Let $B$ be the abelian group generated by the set of polygons in the plane, modulo the subgroup generated by elements $P - \Sigma P_i$, where $P \sqcup P_i$ is a subdivision of a polygon $P$. Any subgroup $G$ of the group of affine motions of the plane acts on $B$. The problem is to compute the quotient group $H_0(G; B)$ of $B$ by the subgroup generated by elements $gb - b$, with $g \in G$, $b \in B$.

1.2 A smooth version. Our purpose is to state and solve a smooth version of the problem. Instead of polygons transforming under affine maps, we consider smooth curves transforming under diffeomorphisms.

The basic tool is a space $M$ (2.1) whose first singular integral homology group $H_1 M$ is a smooth version of the group $B$. Diffeomorphisms of the plane act on $M$ and hence on $H_1 M$. We employ a slight modification of a standard spectral sequence in our calculations.

1.3 Organization. §2 states the key definitions and results; the major proof is in §3. §4 contains the proof of a lemma, and §5 discusses the spectral sequence.

I would like to thank the referee for suggestions and for a simplification in the proof of Lemma 3.5.

2. Results. We require some definitions.

2.1 Definition. Let $M$ be the one-manifold of $C^\infty$ nonsingular curves in $\mathbb{R}^2$, defined as

$$M = \bigsqcup (a, b)_f / \sim$$

where for each $C^\infty$ nonsingular embedding $f$ of an interval $(a, b)$ to $\mathbb{R}^2$ we take a copy $(a, b)_f$ of $(a, b)$, and where if $x \in (a, b)_f$ and $y \in (c, d)_g$ we set $x \sim y$ if and only if there exist neighborhoods $U$ of $x$ in $(a, b)_f$ and $V$ of $y$ in $(c, d)_g$ and a (not necessarily orientation preserving) diffeomorphism $h: U \to V$ such that $f|_U = g \circ h$.

$M$ is a one-dimensional $C^\infty$ nonorientable non-Hausdorff manifold; let $i: M \to \mathbb{R}^2$ denote the obvious immersion. If $g: U \to V$ is a diffeomorphism between open sets in $\mathbb{R}^2$, let $i^*g: i^{-1}U \to i^{-1}V$ denote the resulting diffeomorphism between the open...
subsets $i^{-1}U$ and $i^{-1}V$ of $M$. Let $H_i M$ denote the first singular integral homology group of $M$.

2.2 Definition. Let $H_0 (\Gamma^\infty; H_i M)$ (resp. $H_0 (\Gamma^0; H_i M)$) denote the quotient group of $H_i M$ by the subgroup generated by elements $(i^* g)*b - b$, where $g: U \to V$ is an orientation preserving (resp. area and orientation preserving) $C^\infty$ diffeomorphism between open subsets of $R^2$, and $b \in H_i M$ has support in $i^{-1}U$.

Our problem is to compute the groups just defined.

2.3 Example. The Figure 8 curve (with orientation given by the arrow in Figure 1) defines an element of $H_i M$. Here is one demonstration that this element is 0 in $H_0 (\Gamma^0; H_i M)$. The dotted curve indicates a part of $M$ used in each step.

\[
\begin{align*}
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ \\
\circ & \quad \circ
\end{align*}
\]

$= 0$

Figure 1

2.4 Definition. (i) The **winding maps** $W: H_0 (\Gamma^\infty; H_i M) \to Z$, $W: H_0 (\Gamma^0; H_i M) \to Z$. The tangent line field of $M$ defines a map from $M$ to $R^1$, and hence from $H_i M$ to $H_i R^1$. Picking an isomorphism of $H_i R^1$ with $Z$ gives a map $H_i M \to Z$, which pushes down to the maps $W$.

(ii) The **area map** $A: H_0 (\Gamma^0; H_i M) \to R$: If $b \in H_i M$, let $A(b) = \int_{[b]} x \, dy$ (here $[b]$ denotes the one-current of $R^2$ associated to $b$). $A(b)$ is the “algebraic area enclosed by $b$”. $A$ pushes down to the map $A$.

2.5 Theorem. The maps $W: H_0 (\Gamma^\infty; H_i M) \to Z$ and $W \oplus A: H_0 (\Gamma^0; H_i M) \to Z \oplus R$ are isomorphisms.

2.6 Remark. We compare 2.5 with the classical result. Let $B$ (as in 1.1) be the abelian group generated by polygons in the plane, modulo the subgroup generated by subdivisions. Let $AG1$ and $AS1$ denote the group of orientation preserving affine maps of the plane and the subgroup of area and orientation preserving maps, respectively. Then $[B] H_0 (AG1; B) = 0$, and area gives an isomorphism $A: H_0 (AS1; B) \to R$. There is no “winding map”.

2.7 Remark. If in Definition 2.1 we glue the intervals $(a, b)_t$ together using orientation preserving diffeomorphisms $h$, we obtain a double cover $\tilde{M}$ of $M$, the one-manifold of $C^\infty$ oriented nonsingular curves in $R^2$. There are winding maps $W: H_0 (\Gamma^\infty; H_i \tilde{M}) \to Z$ and $W: H_0 (\Gamma^0; H_i \tilde{M}) \to Z$ defined via the tangent unit vector map from $M$ to $S^1$, and an area map $A: H_0 (\Gamma^0; H_i \tilde{M}) \to R$. One can prove that $W: H_0 (\Gamma^\infty; H_i \tilde{M}) \to Z$ and $W \oplus A: H_0 (\Gamma^0; H_i \tilde{M}) \to Z \oplus R$ are isomorphisms.

3. Proof of 2.5. We shall prove that $W: H_0 (\Gamma^0; H_i M) \to Z \oplus R$ is an isomorphism. The proof for $W: H_0 (\Gamma^\infty; H_i M) \to Z$ is almost identical (see Remark 3.6).

Recall that a topological category is a small category whose sets of objects and morphisms are topologized such that the structure maps of the category are
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continuous. The nerve of a topological category is a simplicial space; we use Segal’s “thick” realization (denoted $|| \cdot ||$ in [9, Appendix A]) to produce a classifying space functor $| \cdot |$ from topological categories to topological spaces.

3.1 Definition. Let $\Gamma^\Omega$ be the topological category whose space of objects is $\mathbb{R}^2$, and whose space of morphisms, denoted $\Gamma^\Omega_i$, is the space of germs of $C^\infty$ area and orientation preserving diffeomorphisms of $\mathbb{R}^2$, with the sheaf topology. Let $D, R: \Gamma^\Omega_i \to \mathbb{R}^2$ denote the domain and range maps of $\Gamma^\Omega$.

The classifying space $| \Gamma^\Omega |$ is the “classifying space for $C^\infty$ codimension 2 foliation, with a transverse orientation and area form”.

3.2 Definition. Let $\Gamma^\Omega \setminus M$ be the topological category of the action $\Gamma^\Omega$ on $M$; the space of objects of $\Gamma^\Omega \setminus M$ is $M$, and the space of morphisms $(\Gamma^\Omega \setminus M)_i$ of $\Gamma^\Omega \setminus M$ is the pullback:

\[
\begin{array}{ccc}
(\Gamma^\Omega \setminus M)_1 & \to & \Gamma^\Omega_i \\
\downarrow D & & \downarrow D \\
M & \to & \mathbb{R}^2
\end{array}
\]

Let $i: \Gamma^\Omega \setminus M \to \Gamma^\Omega$ denote the continuous functor covering the map $i$.

Now we claim [2]

3.3 Proposition. There is a first quadrant spectral sequence $E_{pq}^*$, with differential $d^n$ of bidegree $(-n, n - 1)$, which abuts to $H_{p+1}(\Gamma^\Omega \setminus M)$ and such that $E^{2}_{p0} = H_p|\Gamma^\Omega|$ and $E^{2}_{01} = H_0(\Gamma^\Omega; H_1 M)$.

The spectral sequence is discussed in §5. To apply it to the proof of 2.5 we need two lemmas.

3.4 Lemma [4, 2.6 and 6, Lemma 1]. $H_1|\Gamma^\Omega| = 0$ and $H_2|\Gamma^\Omega| = \mathbb{Z} \oplus \mathbb{R}$.

3.5 Lemma. $H_0|\Gamma^\Omega \setminus M| = \mathbb{Z}/2$.

The proof of 3.5 is in §4.

Proof of Theorem 2.5. Let $K \oplus C: H_2|\Gamma^\Omega| \to \mathbb{Z} \oplus \mathbb{R}$ be the isomorphism of Lemma 3.4. Considering the spectral sequence 3.3, 2.5 will follow from the facts that $A \circ d^2 \circ C^{-1}: \mathbb{R} \to \mathbb{R}$ is an isomorphism and that the image of $W \circ d^2 \circ K^{-1}$ is $2\mathbb{Z}$ (here $d^2$ is the differential for the $E^2$-term). These facts will follow from an explicit description of $d^2: H_2|\Gamma^\Omega| \to H_0(\Gamma^\Omega; H_1 M)$ for elements of $H_2|\Gamma^\Omega|$ represented by closed oriented two-manifolds with an area form.

Let $X$ be such a two-manifold, and let $[X] \in H_2|\Gamma^\Omega|$ be the corresponding homology class; $K[X]$ is the Euler characteristic of $X$, and $C[X]$ is the area of $X$. To describe $d^2[X]$, give a $C^\infty$ cell decomposition $X = \bigsqcup \sigma_i$ of $X$ as in Figure 2. Each cell $\sigma_i$ can be mapped to $\mathbb{R}^2$ by an orientation and area preserving diffeomorphism $f_i$; the boundary of $f_i \sigma_i$, with orientation inherited from $X$, gives a cycle $[\partial f_i \sigma_i] \in H_1 M$. Then $d^2[X] = \sum [\partial f_i \sigma_i]$ is well defined in $H_0(\Gamma^\Omega; H_1 M)$ and independent of the choice of $C^\infty$ cell decomposition of $X$. 

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Clearly \( A \circ d^2 \circ C^{-1} \) is the identity, and a computation with \( X = S^2 \) shows that the image of \( W \circ d^2 \circ K^{-1} \) is 2\( \mathbb{Z} \). This concludes the proof of 2.5.

3.6 Remark. The proof that \( W: H_0(\Gamma^\infty; H_1 M) \to \mathbb{Z} \) is an isomorphism follows §3, except for the substitution of the following lemma for Lemma 3.4.

3.7 Lemma [4, Theorem 3]. \( H_1 |\Gamma^\infty| = 0 \) and \( H_2 |\Gamma^\infty| = \mathbb{Z} \).

4. Proof of 3.5. The real line \( \mathbb{R} \), embedded in \( \mathbb{R}^2 \) as the \( x \)-axis is a submanifold of \( M \). Let \( N \) be the discrete monoid of \( \Gamma^\Omega \backslash M \)-embeddings of the line; as a set 
\[
N = \{ s: \mathbb{R} \to (\Gamma^\Omega \backslash M) \mid D \circ s = \text{id} \text{ and } R \circ s(\mathbb{R}) \subseteq \mathbb{R} \}.
\]

The translates of \( \mathbb{R} \) by \( (\Gamma^\Omega \backslash M)_1 \) generate the topology of \( M \), so by Theorem 1.2(ii) of [1] there is a weak homotopy equivalence \( BN \to |\Gamma^\Omega \backslash M| \). Let us show that \( \pi_1 BN = \mathbb{Z}/2 \).

Let \( K \) be the submonoid of \( N \) consisting of elements which preserve the orientation of the line; it is not hard to see that the exact sequence \( K \to N \to \mathbb{Z}/2 \) gives a homotopy fibration \( BK \to BN \to B\mathbb{Z}/2 \). Since \( \pi_2 B\mathbb{Z}/2 = 0 \), 3.5 will follow when we show that \( \pi_1 BK = 0 \).

So we show that the homomorphic image of \( K \) in any group is trivial. Now \( K \) is generated by elements \( k \) which are the identity section in some open set \( U \) (after [7], 3.1). But for any \( U \) there is an \( m \in K \) such that \( m(\mathbb{R}) \subseteq U \); therefore \( km = m \) and \( k \) must map to the identity of any group. So all of \( K \) must map to the identity.

5. The spectral sequence 3.3. There is a spectral sequence for the action of a pseudogroup on a space, constructed in [2], which generalizes the spectral sequence for the action of a group on a space. The case at hand is an example of its application. We sketch the construction.

Let \( C \) be the discrete category whose objects are contactible open subsets of \( \mathbb{R}^2 \), with morphisms area and orientation preserving embeddings between open sets. Note that (as in [8, §1]) there is a weak homotopy equivalence between \( |C| \) and \( |\Gamma^\Omega| \).

Now recall the immersion \( i: M \to \mathbb{R}^2 \). Let \( S_q \) denote the complex of abelian group valued functors of \( C \), where for \( U \) an open subset of \( \mathbb{R}^2 \), \( S_q U = S_q(i^{-1} U) \), where \( S_q \) is the usual singular \( q \)-chain functor. The spectral sequence for the complex \( S_q \) of functors satisfies 3.3. In particular, \( E^2_{p0} = H_p |\Gamma^\Omega| \) because \( i^{-1} U \) is connected if \( U \) is connected.
Bibliography


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