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SEMIPERFECT FPF RINGS

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ABSTRACT. In this paper we derive some of the structure of semiperfect FPF rings. A ring is right FPF if every f.g. faithful right module is a generator. For semiperfect right and left FPF rings we show that if all one sided zero divisors are two sided zero divisors, then the classical and maximal quotient rings coincide (all four of them) and are self-injective. We show that if the intersection of the powers of the Jacobson radical is zero, then right and left regular elements are regular. Also, we show right FPF semiperfect rings contain the singular submodule of their injective hulls and that every finitely generated module contained in the injective hull and containing the ring is isomorphic to the ring.

Introduction. The objects of study in this paper are semiperfect FPF rings. A ring is right (left) FPF if every finitely generated faithful right (left) module is a generator of the category of all right (left) modules. A ring is FPF if it is both right and left FPF. This notion generalizes the notion of quasi-Frobenius (QF) rings and pseudo-Frobenius (PF) rings. The class of right FPF rings is the class for which a weak version of the fundamental theorem of abelian groups holds, namely: A ring R is right FPF if for every finitely generated faithful right module M, $M^{(n)} \cong R \oplus X$ for some integer n and right module X ($M^{(n)}$, for n an integer, means the direct sum of n copies of M). For an extensive study of semiperfect FPF see Faith **[2, 3, 4]** and for nonsingular FPF rings see Page **[8, 9, 10, 11]**. Faith **[4]** was able to describe commutative FPF rings as those commutative rings for which the maximal quotient ring and classical quotient ring coincide, is self-injective, and all finitely generated faithful ideals are projective. We show in this paper that for a semiperfect FPF ring such that right regular and left regular elements are regular that the right and left maximal and classical quotient rings are all the same and self-injective. This generalizes a theorem of Faith **[3]** for semiperfect Noetherian FPF rings, which states that a Noetherian semiperfect FPF ring is a product of a quasi-Frobenius ring and a semisimple Artinian ring. We also will show under a mild condition that the FPF condition forces right (left) regular elements to be regular. This affords a complete description of a large class of semiperfect FPF rings. Whether this is true in general, i.e. for one sided FPF rings is not known. It is not known if all semiperfect one sided FPF rings are two sided FPF (one sided FPF does not imply two sided FPF in general, nor is the quotient ring self-injective on both sides, see Page **[11]**).

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Throughout this paper all rings will have units and all modules, right or left, will be unitary. For a ring R, $J(R)$ will denote the Jacobson radical. For right (left) R-modules, M_R $({}_RM)$, $Z_r(M_R)$ $(Z_l({}_RM))$ will denote the right (left) singular submodule of M_R $({}_RM)$. The symbol $Q^r_m(R)$ will stand for the maximal right quotient ring of the ring R, and $Q^l_m(R)$ the left maximal quotient ring, $Q^l_{\mathrm{cl}}(R)$ the left classical quotient ring, and Q the right injective hull of R. We will use $x^\perp$ $({}^\perp x)$ to denote the right (left) annihilator of an element x in a right (left) R-module. We will call an element $x \in R$ right (left) regular if $x^\perp = 0$ $({}^\perp x = 0)$.

Semiperfect FPF rings. A ring R is semiperfect if $R/J(R)$ is semisimple Artinian and idempotents lift from $R/J(R)$ to R. We select a complete set of orthogonal primitive idempotents $\{e_{ij}\}^{k,l_i}$, $i = 1,\ldots,k$, $j = 1,\ldots,l_i$ such that $1 = \sum_{i,j}^{k,l_i} e_{ij}$, $e_{ij}R \cong e_{ih}R$ for all j, h; $e_{ij}R \not\cong e_{hf}R$ if $i \neq h$ and for each primitive idempotent e, $eR \cong e_{i1}R$ for some i. If we form $e_0 = \sum_{i=1}^k e_{i1}$, then e_0Re_0 is a ring Morita equivalent to R and is called the basic ring of R. (It is independent, up to isomorphism of the choices made above.) If $R = e_0Re_0$ we say R is self-basic and basic rings are self-basic, Faith [**5**, Chapter 18]. We will maintain the above notation throughout.

A ring is right FPF if every ring Morita equivalent to it is FPF.

The following is found in Faith [**2**].

THEOREM A. *If R is a semiperfect right FPF ring, then for each primitive idempotent e, eR is uniform and if R is self-basic, every nonzero right ideal contains a two sided ideal as a right essential submodule. In particular R has finite uniform, i.e. Goldie dimension.*

THEOREM B. *If R is a semiperfect right FPF ring, then R is right self-injective if every element of $eJ(R)e$ is a left zero divisor, of eRe for every primitive idempotent e.*

One of the key results for semiperfect right FPF rings is that they have a minimal generator of the category finitely generated right R-modules, something like QF-3 rings. Namely, we have,

1.1. THEOREM. *If R is a semiperfect right FPF and M_R is a finitely generated faithful right R-module, then $M_R \cong e_0R \oplus X_R$ for some right R-module X.*

PROOF. Let R be a right FPF semiperfect ring. If M is finitely generated and faithful, M must generate $e_{i1}R$ for each e_{i1}. This says that there is a map λ, of M to $e_{11}R$ with image λ not contained in $e_{ij}J(R)$. Since $e_{11}R/e_{ij}J(R)$ is simple, it follows that $\lambda(m) = e_{11}$ for some m in M. So $M \cong e_{ij}R \oplus M_1$ for some M_1. Now M also generates $e_{21}R$. Since for any map $\gamma\colon M \to e_{21}R$, $\gamma(e_{11}R) \subset e_{21}J$ it must be that M_1 generates $e_{21}R$. Hence $M \cong e_{11}R \oplus e_{21}R \oplus M_2$ for some right R-module M_2. The obvious induction argument finishes the proof.

1.2. COROLLARY. *If R is a self-basic semiperfect ring, then R is right FPF ring iff every faithful finitely generated module M is of the form $R \oplus X$.*

2.1. THEOREM. *If R is a semiperfect right FPF ring and M is a finitely generated submodule of Q which contains R, then $M \cong R$.*

PROOF. Let M be as stated in the hypothesis. Now M is faithful so $M = P \oplus X$ where $0 \neq P$ is projective. We know $P \cong \Sigma_{i=1}^{m} p_i R$ where $p_i R \cong e_j R$ for some e_j. Choose P so that m is maximal. We claim X contains no projective submodules. Suppose not, then there exists an $x \in X$ so that $xR \cong e_{ji} R$ for some e_{ji} where $e_{j1} e_0 = e_{j1}$. Form $X \oplus (e_0 - e_{j1})R = N$. Now N is faithful and for any map of $(e_0 - e_{j1})R$ to $e_{j1}R$, the image is in $e_{j1}J$. It follows that X generates $e_{j1}R$ and hence $X \cong e_{j1}R \oplus Y$ for some Y. This contradicts the maximality of m and establishes the claim. Next write $1 = p + x$ with $p \in P$ and $x \in X$. We claim $x \in Z_r(M_R)$. To see this note that $e_{ij} = pe_{ij} + xe_{ij}$ for each $i = 1, \ldots, k$, and $j = 1, \ldots, l_i$. Now $xe_{ij}R \not\cong e_{ij}R$ and the kernel of the map $R \to xe_{ij}R$ given by left multiplication by xe_{ij} is $(1 - e_{ij})R \oplus W$ where $W \subset e_{ij}J$. Therefore, since $e_{ij}R$ is uniform, this kernel is essential and $xe_i \in Z_r(M)$ for each $i = 1, \ldots, k$, and any j. But $x = \Sigma_{k,j}^{n} xe_{ij}$ so the claim is justified. We now have that $p^{\perp} = 0$ since $x^{\perp}$ is essential. This means $pR \cong R$ and that the uniform dimension of P is the same as that of R. Of course this implies that $X = 0$ and hence that $M = P$ is projective. The next task is to show P is isomorphic to R. To this end we will show $P = N_1 \oplus N_2$ where $N_1 \cong \oplus\Sigma_{j=1}^{l_1} e_{1j}R$ and N_2 is a sum of projective indecomposables none of which is isomorphic to $e_{11}R$. Since P is a generator we know $P \cong e_{11}R \oplus Y$ for some Y. Choose $N_1 = \Sigma_{j=1}^{m_1} p_j R$, $p_j R \cong e_{11}R$, so that $N_1 \oplus N_2 = P$ and N_2 does not contain a summand isomorphic to $e_{11}R$. We want to show $m_1 \geqslant l_1$. Next notice that

$$A = \left(\sum_{\substack{h \neq 1 \\ j}} e_{hj} R \right)^{\perp} \neq 0$$

and is a two ideal contained in $\Sigma_j e_{1j}R$. To see this A cannot be zero for $\Sigma_{h \neq 1, j} e_{hj}R$ cannot generate $e_{i1}R$, by the Krull Schmidt theorem. Also, $A \cap e_{l j_1}R = 0$ for some $e_{1j_1}R$ but $e_{1j_2}R \cap A \neq 0$ violates the fact that $e_{ij_1}R \cong e_{ij_2}R$, so A is essential in $\Sigma_j e_{1j}R$. Now the uniform dimension of A is l_1. Let $1 = n_1 + n_2$ where $n_1 \in N_1$ and $n_2 \in N_2$. $N_2 A = 0$ for N_2 is a sum of projectives isomorphic to summands of $(e_0 - e_{11})R$. This gives $A \subset N_1$ so the uniform dimension $N_1 \geqslant l_1$. It follows that $m_1 \geqslant l_1$. Notice next that N_1 cannot generate $e_{2j}R$, so N_2 must generate $e_{2j}R$. But then, as we just have seen, $N_2 = N_3 \oplus N_4$ where $N_3 \cong \Sigma_{j=1}^{m_2} p_{2j}R$ and $m_2 \geqslant l_2$, and $p_{2j}R \cong e_{21}R$ for all j. The obvious induction now gives $P \cong R \oplus X$, but $X = 0$ by the uniform dimension argument.

2.2. COROLLARY. *If R is a semiperfect right FPF ring, then for each $q \in Q$, $qQ_m^r(R) + Q_m^r(R) \cong Q_m^r(R)$.*

PROOF. Trivially $qR + R$ is dense in $qQ_m^r(R) + Q_m^r(R)$. But the isomorphism $qR + R \cong R$ lifts to a $Q_m^r(R)$ isomorphism of Q. Also under this extended isomorphism q is sent to an element of R hence of $Q_m^r(R)$ ar.d one is also in the image so that the image of $qQ_m^r(R) + Q_m^r(R)$ is $Q_m^r(R)$.

2.3. COROLLARY. *Let R be as in Theorem 2.1. Let $x \in Z_r(Q)$. Then $x \in R$.*

PROOF. Let x be as stated. We have that $xR + R = dR$ for some $d \in Q$ with $d^{\perp} = 0$. Now $d = r_1 + xr_2$ and there is an r_3 such that $dr_3 = x$. Since $d^{\perp} = 0$, $r_3^{\perp} = x^{\perp}$ is essential and $r_3 \in Z_r(R)$ which is contained in J since $Z_r(R)$ contains no idempotents. So $x = r_1r_3 + xr_2r_3$ or $x(1 - r_2r_3) = r_1r_3$ and $x = r_1r_3(1 - r_2r_3)^{-1} \in R$. Faith [**3**] calls such rings with $J(Q_m^r(R)) \subset R$ sandwich rings.

2.4. COROLLARY. *If R is right FPF semiperfect and $Q = Q_m^r(R)$, then $J(R) \supset J(Q) = Z_r(R) = Z_r(Q)$.*

PROOF. For a right self-injective ring we have $Z_r(Q) = J(Q)$ by Utumi [**13**, Lemma 4.1].

The next lemma points out the importance of having right regular elements left regular.

3. THEOREM. *Let R be a semiperfect right FPF ring. If all right regular elements are left regular, then the regular elements are units in $Q_m^r(R)$.*

PROOF. Let Q be the right injective hull of R. Let $\Lambda = \mathrm{Hom}_R(Q, Q)$. Then Λ is a Dedekind finite ring, i.e. $xy = 1 \to yx = 1$ in Λ, since it has no infinite sets of orthogonal idempotents, Jacobson [**6**]. Let $r \in R$ be such that $r^{\perp} = 0$. Then the map $x \to rx$ induces an isomorphism of Q, i.e. a unit of Λ which we will denote by r also. So for some $\lambda \in \Lambda$, we have $\lambda \circ r = r \circ \lambda = 1$, i.e. $\lambda(r) = \lambda(1)r = 1$. Now take $\theta \in \Lambda$ such that $\theta(1) = 0$. To show that $\lambda(1)$ is in $Q_m^r(R)$, we must show that $\theta(\lambda(1)) = 0$, see Lambeck [**7**, Proposition 1, p. 94]. We have that $\theta(\lambda(1))r = 0$. Let $\theta(\lambda(1)) = y$. Form $yR = R$. Let the embedding of $yR + R$ into R be given by f. Then $f(y)$ is in R and $f(y)r = 0$, so $f(y) = 0$, i.e. $f(\theta(\lambda(1))) = 0$ so that $\theta(\lambda(1)) = 0$. Hence $\lambda(1)$ is $Q_m^r(R)$ and r is a unit in $Q_m^r(R)$ which is also Dedekind finite for the same reason Λ was.

4.1. PROPOSITION. *If R is a semiperfect FPF ring with right regular elements left regular, then $Q_m^r(R) = Q_{\mathrm{cl}}^l(R) = Q$, the right injective hull of R.*

PROOF. We wish to show every element of Q is of the form $\alpha^{-1}b$ for some α and b in R with α regular. Let $q \in Q$, then $qR + R \subset dR$ for some d in Q with $d^{\perp} = 0$. But $dR \supset R$ so there exists an element α in R with $d\alpha = 1$. Now α is right regular so $d = \alpha^{-1} \in Q_m^r(R)$ hence $q \in Q_m^r(R)$. Also, $q = \alpha^{-1}b$ for some b in R which completes the proof.

5. THEOREM. *If R is a semiperfect FPF (both sides) ring, with right regular and left regular elements regular, then $Q_m^r(R) = Q_m^l(R) = Q_{\mathrm{cl}}^r(R) = Q_{\mathrm{cl}}^l(R) = Q$ and Q is a right and left self-injective.*

PROOF. This follows directly from the right and left hand versions of Theorem 3.

REMARK 6. For any ring, R, with no infinite sets of orthogonal idempotents with right injective $Q_m^r(R)$ right regular elements must be left regular so the conditions on regularity are clearly necessary in order that the maximal quotient rings be injective.

Next we show that many semiperfect right FPF rings do have right and left regular elements regular.

7.1. LEMMA. *Let R be a semiperfect right FPF ring. Let d be a right regular element of R. Then $\bigcap_{n=1}^{\infty} Rd^n \supset {}^{\perp}d$.*

PROOF. Let $xd = 0$, and form $F = R \otimes R$. Let $M = (d, x)R$ and consider $F/M = N$. We claim N is faithful. To see this suppose $(0,1)r \in M$. Then $(0,1) = (d, x)r_0$ for some r_0. But then $dr_0 = 0$, so $r_0 = 0$ and $xr_0 = r = 0$. This also shows $(0,1)R \cap M = 0$. Now $(0,1)R \cong R$ so there is a map f of N into Q such that $f(0,1)R = R$. N is a finitely generated submodule of Q which contains R. By Theorem 2.1 we have an epimorphism γ of N onto R. Now let $\gamma(1,0) = r_1$ and $\gamma(0,1) = r_2$. Then $R = r_1R + r_2R$. We claim $r_1 \in Z_r(R) \subset J$. We have that $(d^2, 0)R \subset M$ for $(d^2, 0) = (d, x)d$. This means $r_1 d^2 = 0$. But since $(d^2)^{\perp} = 0$, $R \cong d^2R$. Also d^2R is right essential in R because the uniform dimension of d^2R is the same as that of R. This gives $r_1 \in Z_r(R)$ so, since r_1R is small in R, $r_2R = R$, hence r_2 is a unit. Now $r_1d + r_2x = 0$ so $-r_2^{-1}r_1d = x$ and $x \in Rd$. We may repeat the above to $M_n = (d^n, x)R$ and $N_n = F/M_n$ for any n and hence that $x \in Rd^n$ for all n.

7.2. THEOREM. *Let R be a semiperfect right and left FPF ring. If for each idempotent e, and element $d \in eJe$ such that $d^{\perp} \cap eRe = 0$, we have $\bigcap_{n=1}^{\infty} Rd^n = 0$, then right regular elements are regular.*

PROOF. If d is right regular and d is right regular modulo the radical J, then d is a unit. So we can assume d is not right regular modulo J. It follows that there is an idempotent e, so that $de \in J$. Now for primitive idempotents f and g, with $fR \cong gR$, if $df \in J$ then $dg \in J$ too. This means we can take the idempotent e so that $de \in J$ and if f is a primitive idempotent with $f = (1-e)f = f(1-e)$, then $df \notin J$, and if g is a primitive idempotent with $ge = eg = g$ $gR \not\cong fR$. This last statement implies that $ed(1-e)$ and $(1-e)de$ are in $Z_r(R) \cap Z_l(R)$. Now we have

$$d = ede + (1-e)d(1-e) + z$$

with $z \in Z_r(R) \cap Z_l(R)$. It follows that $ede + (1-e)d(1-e)$ is a right regular element. It is easy to see that $(1-e)d(1-e)$ is right regular in

$$(1-e)R(1-e) \text{ modulo } (1-e)J(1-e)$$

since $ed(1-e) \in J$. So $(1-e)d(1-e)$ is a unit in $(1-e)R(1-e)$. Now apply Lemma 7.1 to the right regular element $ede + (1-e)d(1-e)$ and any y such that $y(ede + (1-e)d(1-e)) = 0$. This says

$$\begin{aligned} y &\in \bigcap_{n=1}^{\infty} R(ede + (1-e)d(1-e))^n \\ &= \bigcap_{n=1}^{\infty} R\big((ede)^n + ((1-e)d(1-e))^n\big) \\ &= \bigcap_{n=1}^{\infty} R((1-e)d(1-e))^n, \end{aligned}$$

in particular that $ye = 0$. But for $y = y(1-e)$, $y(1-e)d(1-e)u = y(1-e)$ for some $u \in (1-e)R(1-e)$ and hence that $y = 0$. We now have that $ede + (1-e)d(1-e)$ is regular. But since $z \in Z_l(R)$, it follows that d is left regular hence regular.

7.3. Corollary. *If R is right FPF and has a.c.c. on left annihilators, then right regular implies left regular.*

Proof. Let $d^{\perp}=0$ such that ${}^{\perp}d$ is maximal. Then Lemma 7.1 implies that if $yd=0$, then $y=rd$ some $r\in R$. But then $rd^2=0$ and since ${}^{\perp}d$ is maximal ${}^{\perp}d={}^{\perp}(d^2)$ and hence $rd=y=0$.

7.4. Corollary. *If R is left Noetherian right FPF, then R is a left order in a quasi-Frobenius ring.*

Proof. By Proposition 4.1 and Corollary 7.3 $Q^l_{\mathrm{cl}}(R)=Q^r_m(R)$. This implies $Q^l_{\mathrm{cl}}(R)$ is right self-injective and left Noetherian and therefore quasi-Frobenius.

7.5. Corollary. *A Noetherian semiperfect FPF ring is a product of Dedekind prime rings and quasi-Frobenius rings.*

Proof. By [**11**] we can decompose an FPF ring into a product $R_1\times R_2$ with R_1 semiprime and R_2 with essential left or right nil ideal. By Theorem 10A of [**3**] R_1 is Dedekind. By Corollaries 2.3 and 7.4 and Theorem 2 of [**1**] R_2 is Artinian. But Artinian FPF rings are quasi-Frobenius.

7.6. Corollary. *If R is right and left FPF and semiperfect, with $\bigcap_{n=1}^{\infty}Rd_1^n=0=\bigcap_{n=1}^{\infty}d_2^nR$ for all right regular d_1 and left regular d_2, of eJe in eRe for any idempotent then $Q^r_m(R)=Q^l_m(R)=Q^l_{\mathrm{cl}}(R)=Q^r_{\mathrm{cl}}(R)=Q$.*

Proof. This just combines Theorems 5 and 7.2. We obtain a partial converse namely,

8. Theorem. *Let R be a semiperfect ring, with* (i) $Q^r_m(R)=Q^r_{\mathrm{cl}}(R)=Q=Q^l_{\mathrm{cl}}(R)$, (ii) $Q^r_m(R)$ *is right FPF*, (iii) e_0Re_0 *is strongly bounded*, (iv) *every finitely generated right ideal of R which contains a regular element is a generator. Then R is right FPF.*

Proof. Let M be a finitely generated faithful right R-module. We wish to show $M\otimes_R Q$ is a faithful Q-module. It is easy to see $M\otimes Re_0=N$ is a faithful e_0Re_0-module. Now let $\{n_i\}_{i=1}^k$ generate N over e_0Re_0. Since e_0Re_0 is strongly bounded $\bigcap n_i^{\perp}=0$. So e_0Re_0 embeds in $N^{(m)}$ for some m. We have by [**12**, Proposition 3.2, p. 219] that e_0Qe_0 is the maximal right ring of quotient of e_0Re_0. We claim e_0Qe_0 is left flat over e_0Re_0. To see this, we know Q is left flat over R because $Q=Q^r_{\mathrm{cl}}(R)$ so $IQ=I\otimes Q$ for all right ideals I of R. Now each right ideal H of e_0Re_0, is of the form $H=e_0Ie_0$ for a right ideal I of R and

$$
\begin{aligned}
H\otimes_{e_0Qe_0}e_0Qe_0 &\cong e_0Ie_0\otimes e_0Qe_0\cong e_0Ie_0\otimes_{e_0Re_0}e_0Q\otimes_{e_0Qe_0}Qe_0\\
&\cong\left(e_0Ie_0\otimes_{e_0Re_0}e_0R\right)\otimes_R\left(Q\otimes_{e_0Qe_0}Qe_0\right)\cong\left(e_0Ie_0\otimes e_0R\otimes_R Q\right)\otimes Qe_0\\
&\cong\left(e_0I\otimes_R Q\right)\otimes_{e_0Qe_0}Qe_0\cong\left(e_0IQ\right)\otimes_{e_0Qe_0}Qe_0\cong e_0IQe_0
\end{aligned}
$$

so that e_0Qe_0 is left flat over e_0Re_0. We have an exact sequence $0\to Re_0\to N^{(m)}$. Tensoring this with e_pQe_0 over e_0Re_0 gives

$$0\to e_0Qe_0\to N^{(m)}\otimes e_0Qe_0\cong\left(M\otimes Re_0\right)^m\otimes e_pQe_0.$$

So $(M \otimes Re_0) \otimes e_0Qe_0$ is a faithful e_0Qe_0-module. But then $M \otimes_R Q$ is a faithful Q-module. Now $M \otimes_R Q$ must generate Q. So there are maps f_i: $M \otimes_R Q \to Q$ so that $\Sigma_{i,j} f_i(m_j \otimes q_{ij}) = 1$. We have that the image of M in $M \otimes Q$ generates $M \otimes Q$. And we can take the $\{m_j\}$ to generate M. Letting $f_i(m_j \otimes 1) = b_{ij}^{-1} a_{ij}$ and $q_{ij} = c_{ij} d_{ij}^{-1}$ we can find regular b and d so that $bf_i(m_j \otimes 1) \in R$ for all i and j and $q_{ij}d \in R$ for all i and j. Then each bf_i restricted to the image of M in $M \otimes Q$ gives a map of M into R and $bd = \Sigma\, bf_i(m_j \otimes q_{ij}d)$ so bd is in the trace of M in R. By condition (iv) M is a generator.

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