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MONOMIAL EQUIMULTIPLE CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. It is known that the local equimultiple locus of a hypersurface in characteristic zero is contained in a regular hypersurface. Here we give an example of a monomial curve on a threefold in positive characteristic $p > 2$ which is equimultiple but not hyperplanar. As a corollary we have that any monomial curve which lies on a certain type of hypersurface (whose local equation is of a special form in its natural p-basis expression) is automatically equimultiple for the hypersurface.

Introduction. Consider the hypersurface $F \in k[[X_1,\ldots,X_n, Z]]$ whose local equation is

$$F = Z^d + a_1 Z^{d-1} + \cdots + a_d$$

such that $a_i \in k[[X_1,\ldots,X_n]]$ and $\operatorname{ord} F = d$. Let P be a prime ideal in $k[[X_1,\ldots,X_n, Z]]$ such that $F \in P^{(d)}$ (the dth symbolic power of P). Then since

$$DF \in P^{(d-1)} \text{ for any derivation } D \text{ on } k[[X_1,\ldots,X_n, Z]],$$

taking $(d-1)$ partial derivatives with respect to Z we get

$$D_Z^{(d-1)}(F) = d!Z + a_1(d-1)! \in P.$$

If $\operatorname{char} k = 0$, this may be interpreted as saying that the equimultiple curve P is contained in the hyperplane (i.e. regular hypersurface) $Z + a_1/d = 0$. (See Abhyankar's paper [**A**] for details.)

In positive characteristic it is known that any equimultiple curve on a surface is hyperplanar. [Some cases of this were done in an earlier paper [**N**]; a general 'characteristic-free' proof was given recently by S. B. Mulay (to appear).] In [**N**] we gave an example *in characteristic two* showing that an equimultiple curve on a variety of dimension bigger than two need not be hyperplanar. In this paper we give an example of a (3-dimensional) variety in characteristic $p < 2$ having an equimultiple curve which is not hyperplanar.

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The main aim of this paper is to prove the following

THEOREM. *Let k be an algebraically closed field of characteristic $p > 2$. Let $A = k[[X, Y, Z, W]]$, and let $P = \ker \phi$ where ϕ is given by*

$$A \overset{\phi}{\to} k[[t]] \qquad (t \text{ an indeterminate}),$$

$$\begin{aligned} X &\mapsto t^{a_1} \quad \text{with } a_1 = (2p-2)(2p+1), \\ Y &\mapsto t^{a_2} \qquad a_2 = p(2p+1), \\ Z &\mapsto t^{a_3} \qquad a_3 = p(2p-1), \\ W &\mapsto t^{a_4} \qquad a_4 = c-1, \end{aligned}$$

where c is the conductor of the semigroup $\langle a_1, a_2, a_3 \rangle$. Then there exists a hypersurface defined by

$$F = W^p + \sum_{\substack{0 \leq i, j \leq p-1 \\ (i, j) \neq (0,0)}} \left(f_{ij}(X, Y, Z)\right)^p Y^i Z^j$$

such that

(i) *P is an equimultiple curve on F, i.e.*, $\operatorname{ord}_A F = \operatorname{ord}_{A_P} FA_P = p$;

(ii) *P does not contain a regular parameter of A.*

The proof will follow from the following lemmas.

LEMMA 1. *With the same notation as above, there exist f_{ij} for all i, j with $0 \leq i, j \leq p-1$, $(i, j) \neq (0,0)$ satisfying the following conditions*:

(a) $\operatorname{ord}_t(\phi(f_{ij})) \in \langle a_1, a_2, a_3 \rangle$,

(b) $f_{ij} \neq 0$,

(c) $\phi(F)$ *is homogeneous* (*in* $k[t]$).

(REMARK. (a) implies the f_{ij}'s can, in fact, be chosen to be monomials in X, Y, Z.)

PROOF. Let $Q = P \cap k[[X, Y, Z]]$. Let x, y, z denote the residues of X, Y, Z modulo P. We note that the numbers a_1, a_2, a_3 satisfy the following conditions:

(i) $(a_1, a_2, a_3) = 1$. (In fact, since $a_1 = 2a_3 - 2$ and a_3 is odd, $(a_1, a_3) = 1$.)

(ii) $[(a_1, \ldots, a_i), a_{i+1}] \in \langle a_1, \ldots, a_i \rangle$ for $i = 1, 2$. (As usual, $(a_1, \ldots, a_n)$ denotes the g.c.d., and $[a_1, \ldots, a_n]$ denotes the l.c.m. of $a_1, \ldots, a_n$.)

Hence we may apply Proposition 2.1 of [**H**] to conclude that:

(i)′ Q is an ideal-theoretic complete intersection. (This also follows by observing that $X^p - Y^{2p-2}$ is a minimal relation in Q in the sense of [**H**].) Hence

(ii)′ The semigroup $\langle a_1, a_2, a_3 \rangle$ is symmetric.

(iii)′ The conductor c of S is given by

$$\begin{aligned} a_4 = c - 1 &= \sum_{i=1,2} [(a_1, \ldots, a_i), a_{i+1}] - \sum_{i=1}^{3} a_i \\ &= 8p^3 - 10p^2 - p + 2. \end{aligned}$$

Now the condition for F to be weighted homogeneous is

$$pa_4 = p\left[\operatorname{ord}_t \phi(f_{ij})\right] + ia_2 + ja_3,$$

i.e.

$$a_4 = \operatorname{ord}_t \phi(f_{ij}) + i(2p+1) + j(2p-1). \tag{I}$$

Hence to prove the existence of the f_{ij}'s, using (ii)′ and (iii)′ above, we need only prove the following.

Claim. $i(2p+1) + j(2p-1) \notin S$ for $0 \leqslant i,\ j \leqslant p-1$, $(i, j) \neq (0,0)$.

PROOF OF CLAIM. Suppose, if possible,

$$i(2p+1) + j(2p-1) = \lambda a_1 + \mu a_2 + \nu a_3, \qquad \lambda, \mu, \nu \geqslant 0.$$

Then

$$\max_{0 \leqslant i,j \leqslant p-1} i(2p+1) + j(2p-1) = (p-1)(4p) < 4p^2 - 2p - 2 = a_1$$

gives $\lambda = 0$. Thus

$$i(2p+1)j(2p-1) = \mu p(2p+1) + \nu p(2p-1) \Rightarrow p \mid (i-j).$$

But $0 \leqslant i, j \leqslant p-1 \Rightarrow |i-j| \leqslant p-1$. Thus $i = j \neq 0$ and

$$2i + 2j = \mu(2p+1) + \nu(2p-1) \Rightarrow 4i = \mu(2p+1) + \nu(2p-1), \qquad i \leqslant p-1.$$

Clearly $(\mu, \nu) = (1,0)$ or $(0,1)$ is impossible and $\mu \geqslant 1$ or $\nu \geqslant 1 \Rightarrow \mu(2p+1) + \nu(2p-1) \geqslant 4p-2$, contradiction. This proves the Claim, and since, in (I),

$$a_4 - \left[\max_{0 \leqslant i,j \leqslant p-1} i(2p+1) + j(2p-1)\right] \geqslant a_4 = p(4p),$$
$$> 0 \quad \text{when } p \geqslant 2 \text{ by (iii)}',$$

we get $f_{ij} \neq 0$ for all the relevant i and j. This completes the proof of Lemma 1.

LEMMA 2. *Let* $A = k[[X, Y, Z, W]]$. *Let* $P \in \operatorname{Spec} A$ *be such that* $\dim A/P = 1$; *let* $k[[t]]$ *denote the integral closure of* A/P *in its quotient field. Let* x, y, z *and* w *denote the residues of* X, Y, Z *and* W *modulo* P. *Assume* $x = t^{a_1}$, $y = t^{a_2}$, $z = t^{a_3}$, $w = t^{a_4}$ *for some positive integers* a_1, a_2, a_3, a_4. *Let* $F = W^p + f(X, Y, Z)$, *where*

$$f(X,Y,Z) = \sum_{\substack{0 \leqslant i,j \leqslant p-1 \\ (i,j) \neq (0,0)}} f_{ij}^p Y^i Z^j$$

such that

(a) *the* f_{ij} *are nonzero monomials in* X, Y, Z *and* $f_{ij}(1,1,1) = 1$,

(b) F *is weighted homogeneous* (*with weights* a_1, a_2, a_3, a_4 *for* X, Y, Z *and* W).

Then $F \in P^{(p)}$.

PROOF. Let D_{ij} denote the (usual) partial derivatives with respect to Y and Z. Then $F \in P^{(p)} \Leftrightarrow D_{ij}(F) \in P$ for $0 \leqslant i + j \leqslant p-1$. (For a proof of this, see Lemma 4.3.3(i) of [G].) By hypothesis (conditions (a) and (b)) P lies on F, i.e. $D_{00}(F) \in P$.

Now $F(X, Y, Z, W)$ is weighted homogeneous. Hence all derivatives $D_{ij}(F)$ are again weighted homogeneous. (This observation is independent of the special forms of F and f_{ij}'s.) Hence

$D_{ij}(F) \in P$ for $0 < i + j \leqslant p-1 \Leftrightarrow$ the sum of the coefficients of $D_{ij}(f)$

(in the given form, i.e., in the expansion of $D_{ij}(f)$ with respect to the usual p-basis $\{Y^iZ^j\}_{0 \leq i,j \leq p-1}$) equals zero (mod p).

But to prove this statement we may further assume $f_{ij} = 1$ for $0 \leq i, j \leq p-1$, $(i, j) \neq (0,0)$, i.e.

$$f = \sum_{\substack{0 \leq i,j \leq p-1 \\ (i,j) \neq (0,0)}} Y^iZ^j = \left(1 + Y + \cdots + Y^{p-1}\right)\left(1 + Z + \cdots + Z^{p-1}\right) - 1$$

$$= (1-Y)^p(1-Z)^p(1-Y)^{-1}(1-Z)^{-1} - 1.$$

Now, the sum of the coefficients of $D_{ij}(f)$

$$= D_{ij}(f)\big|_{\text{evaluated at } (Y,Z)=(1,1)}$$

and this is obviously zero for

$$\text{(II)} \qquad 0 < i + j < 2p - 2$$

This proves Lemma 2.

Proof of the Theorem. Using Lemma 1 we choose $f_{ij}(X, Y, Z)$ to be monomials in X, Y, Z satisfying the condition $f_{ij}(1, 1, 1) = 1$ (besides the conditions of the lemma). Then Lemma 2 shows that P is an equimultiple curve on F. To see that P is not contained in a regular hypersurface we only have to observe that, as in **[N]**, among the four integers a_1, a_2, a_3, a_4, none is contained in the semigroup generated by the other three. Q.E.D.

Remarks. Lemma 2 is easily seen to be equivalent to a 'binomial identity' in characteristic $p > 0$. The advantage of our formulation and method of proof is that it extends to several variables. Thus we have the following general property of monomial equimultiple curves on hypersurfaces in higher dimensions:

Proposition. *Let $A = k[[X_1, \ldots, X_n, W]]$ with $n \geq 2$. Let $P \in \operatorname{Spec} A$ be such that* dim $A/P = 1$; *let $k[[t]]$ denote the integral closure of A/P in its quotient field. Let x_i and w denote the residues of X_i and W modulo P, for $i = 1, 2, \ldots, n$.*

Let $F = W^p + f(X_1, \ldots, X_n)$ where

$$f(X_1, \ldots, X_n) = \sum_{0 < i \ll p} f_i^p X^i$$

with $X \subseteq (X_1, \ldots, X_n)$, card$(X) \geq 2$ *and $0 < i \ll p$ denotes, as usual, a restricted vector $(i_1, \ldots, i_m)$,*

$$0 \leq i_j \leq p - 1 \quad \text{for } j = 1, 2, \ldots, m, \qquad (i_1, \ldots, i_m) \neq (0, \ldots, 0).$$

Assume:

(a) *$x_i = t^{a_i}$, $w = t^{\omega}$ for some positive integers a_i, ω, $i = 1, 2, \ldots, n$.*

(b) *f_i's are nonzero monomials in $X_1, \ldots, X_n$ with coefficient* 1.

(c) *P lies on F.*

Then P is equimultiple for F.

Proof. The proof of Lemma 2 obviously generalises to this situation; observe that the restriction card$(X) \geq 2$ is essential for the validity of the final step (II) in the proof.

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