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A GLOSS ON A THEOREM OF FURSTENBERG

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ABSTRACT. Certain refinements are offered for Furstenberg's ergodic-theoretic version of Szemeredi's theorem.

Furstenberg [**1977**] has proven a significant generalization of a theorem of Poincaré, which, with no real loss, can be formulated thus: If k is a positive integer and B_1, $B_2, \ldots$ is a stationary sequence of events of positive probability in a countably additive probability space, then there is a k-progression, K, such that $B_K = \bigcap B_i\,(k \in K)$ has positive probability. (A *k-progression* is a set of k integers of the form $\{a, a+b, a+2b, \ldots, a+(k-1)b\}$ with $a \geqslant 0$, $b > 0$.)

The present paper observes that neither the hypothesis of countable additivity nor of stationarity is needed. Moreover, the probability of B_K can be bounded from below by a $\delta > 0$ which depends only on k and $p = P(B_1)$. These facts are immediate corollaries to:

THEOREM 1. *Let $p > 0$ and let k be a positive integer. Then there is a $\delta > 0$ and a positive integer n such that, for every n-tuple of events $B_1, \ldots, B_n$ of average probability at least p, there is a k-progression $K \subset \{1, \ldots, n\}$ for which $\bigcap B_i$ $(i \in K)$ has probability at least δ.*

This form of Furstenberg's theorem follows by an argument which he chose not to provide in [**1977**]. Indeed, it is a simple consequence of Szemeredi's theorem [**1975**] on the existence of arbitrarily long arithmetic sequences in each set of integers of positive density. But it is convenient first to provide a trivial lemma.

LEMMA 1. *Let $B_1, \ldots, B_n$ be events of average probability at least p and let l be a positive integer less than n. Then there is a subset X of $\{1, \ldots, n\}$ of cardinality l such that*

$$P\Big(\bigcap B_i\,(i \in X)\Big) \geqslant \Big(p - \frac{l}{n}\Big) \Big/ \binom{n}{l}. \tag{1}$$

PROOF OF LEMMA 1. Let Y be the number of B that occur. Since Y is at most n on the event $(Y \geqslant l)$ and is at most $l-1$ on its complement, the following inequality (sharp) is easily obtained.

$$P(Y \geqslant l) \geqslant \Big(\frac{PY}{n} - \frac{l-1}{n}\Big)\Big(1 - \frac{l-1}{n}\Big)^{-1}. \tag{2}$$

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(In (2), the precision (expectation) of Y is designated by PY as accords with a notational innovation of de Finetti.)

For the purposes of this note, this weaker inequality suffices:

$$P(Y \geqslant l) \geqslant \frac{PY}{n} - \frac{l}{n}. \tag{3}$$

Plainly, the event $Y \geqslant l$ is the union of the events $\bigcap B_i$ $(i \in X)$ as X ranges over $[n]^l$, the subsets of $\{1, \ldots, n\}$ of cardinality l. Therefore,

$$P(Y \geqslant l) \leqslant \sum P\Big(\bigcap B_i\,(i \in X)\Big) \leqslant \binom{n}{l} \max P\Big(\bigcap B_i\,(i \in X)\Big), \tag{4}$$

as X ranges over $[n]^l$. So, for some $X \in [n]^l$,

$$\begin{aligned} P\Big(\bigcap B_i\,(i \in X)\Big) &\geqslant P(Y \geqslant l)\Big/\binom{n}{l} \geqslant \left(\frac{PY}{n} - \frac{l}{n}\right)\Big/\binom{n}{l} \\ &\geqslant \left(p - \frac{l}{n}\right)\Big/\binom{n}{l}, \end{aligned} \tag{5}$$

where the second inequality obtains in view of (3), and the third by hypothesis. □

Let $\gamma_k(n)$ be the least integer l such that, if X is a subset of $\{1, \ldots, n\}$ of cardinality l, then X includes a k-progression. Szemeredi **[1975]** has shown that $\gamma_k(n)/n$ converges to 0 as $n \to \infty$.

PROOF OF THEOREM 1. By Szemeredi's theorem, there is an $n = n(p, k)$ such that $\gamma_k(n) < np/2$. For $l = \gamma_k(n)$, let δ be $p/2\binom{n}{l}$. That (δ, n) satisfies Theorem 1 can be verified, thus. Let $B_1, \ldots, B_n$ be events of average possibility at least p. By Lemma 1, there is an $X \subset \{1, \ldots, n\}$ of cardinality l such that (1) holds. Since $l/n < p/2$, the right-hand side of (1) is at least δ. So $\bigcap B_i$ $(i \in X)$ has probability no less than δ. Since X is of cardinality $\gamma_k(n)$, X includes a k-progression, K. Plainly, $\bigcap B_i$ $(i \in K)$ includes $\bigcap B_i$ $(i \in X)$. So it, too, has probability no less than δ. □

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