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THE FREIHEITSSATZ FOR ONE-RELATION MONOIDS[1]

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ABSTRACT. We give an elementary proof of the Freiheitssatz for one-relation monoids.

The Freiheitssatz is basic to the study of one-relator groups. It states that for a group G presented by generators X and single (cyclically reduced) defining relator r, if $Y \subseteq X$ excludes some generator occurring in r then the subgroup of G generated by Y is freely generated by Y. The analogous property is true of one-relation monoids, that is, of Thue systems with a single rule. A proof of this fact by appeal to the Freiheitssatz for groups [**4**, **5**, **3**] is possible; it can also be derived from the theorem of Gerstenhaber and Rothaus on solutions of nonsingular sets of equations over residually finite groups [**1**]. We give here a direct proof, based on a construction used by Levin [**2**], for which only elementary knowledge of groups and monoids is required.

For an alphabet (set of symbols) Σ, Σ^* denotes the free monoid with generators Σ, with the identity denoted by e.

A Thue system with a single rule is a set $T = \{(u,v)\}$ consisting of a pair of words over an alphabet Σ. The congruence $\overset{*}{\leftrightarrow}$ on Σ^* associated with such a system T is defined as follows: for any strings $x, y \in \Sigma^*$, define $xuy \leftrightarrow xvy$, and define $\overset{*}{\leftrightarrow}$ to be the reflexive, symmetric and transitive closure of $\leftrightarrow$. The quotient $\Sigma^*/\overset{*}{\leftrightarrow}$ of Σ^* by the congruence is a monoid, the monoid presented by $\langle \Sigma | u = v \rangle$.

THEOREM. *Let Σ be an alphabet, Γ a subset of Σ, and u, v strings in Σ^*. Consider the Thue system $\{(u,v)\}$ with associated congruence $\overset{*}{\leftrightarrow}$ and monoid $M = \Sigma^*/\overset{*}{\leftrightarrow}$. If a letter not in Γ occurs in u or v then for any $x, y \in \Gamma^*$, $x \overset{*}{\leftrightarrow} y$ implies $x = y$. In other words, if $uv \notin \Gamma^*$ then the submonoid of M generated by the congruence classes of Γ is freely generated by them.*

PROOF. If both u and v contain letters not in Γ, then the rule $u \leftrightarrow v$ does not apply to any word in Γ^* and the conclusion of the theorem is clearly true. Suppose, therefore, that u, but not v, contains letters not in Γ.

We may assume that Γ includes all the letters of Σ except one, say a. Suppose u has some $n > 0$ occurrences of a and write u as $wau_0 \cdots au_{n-1}$, where w and each u_i are in Γ^*.

To establish the theorem, it is sufficient to prove that there is a group G and a homomorphism $\phi\colon M \to G$ that is one-to-one on (the congruence classes of) Γ such that $\phi(\Gamma)$ freely generates a free subgroup of G. Let F be the free group on Γ, with

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identity 1_F. View the rule $u \leftrightarrow v$ as an equation "$au_0 \cdots u_{n-2}a(u_{n-1}v^{-1}w) = 1$" to be solved for the variable a over F: applying Levin's theorem [**2**], there is a group G containing F as a subgroup and an element $\overline{a}$ of G such that $\overline{a}u_0 \cdots \overline{a}u_{n-1}v^{-1}w = 1_G$. The homomorphism ϕ can then be defined via the inclusion of Γ^* in F with $\phi(a) = \overline{a}$, thus completing the proof.

For the Thue system $\{(u,v)\}$, Levin's construction takes the following form. The group G is the wreath product of F with the cyclic group $Z_n = \{0, 1, \ldots, n-1\}$: that is, elements of G are pairs (k, C) with $k \in Z_n$ and $C\colon Z_n \to F$ an arbitrary function; and multiplication is given by $(m, C_1)\cdot(k, C_2) = (m+k, C_3)$ where $C_3(i) = C_1(i-k)C_2(i)$, $0 \le i \le n-1$, and the index computation is modulo n.

For $x \in \Gamma^*$ define $\hat{x}\colon Z_n \to F$ by $\hat{x}(i) = x$, $0 \le i \le n-1$. Let $A(i) = u_i^{-1}$ for $0 \le i \le n-2$ and let $A(n-1) = w^{-1}vu_{n-1}^{-1}$. Finally, let $h\colon \Sigma^* \to G$ be the homomorphism determined by defining $h(a) = (-1, A)$ and, for $b \in \Gamma$, $h(b) = (0, \hat{b})$. Note that for $x \in \Gamma^*$, $h(x) = (0, \hat{x})$. (In the case $n = 1$, this reduces to the homomorphism $h\colon \Sigma^* \to F$ given by $h(a) = w^{-1}vu_0^{-1}$, $h(b) = b$.)

It follows from the definition of h that $h(u) = h(v)$, and therefore for any x, $y \in \Sigma^*$, if $x \overset{*}{\leftrightarrow} y$ then $h(x) = h(y)$. In particular, if x, $y \in \Gamma^*$ and $x \overset{*}{\leftrightarrow} y$ then from $h(x) = h(y)$ we conclude that $(0, \hat{x}) = (0, \hat{y})$ and so $x = y$, as desired. □

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