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# THE ASYMPTOTIC NORMING PROPERTY AND MARTINGALE CONVERGENCE

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ABSTRACT. A martingale proof is given of the result of R. G. James and A. Ho in [3] that the asymptotic norming property implies the Radon-Nikodym property.

In [3] R. C. James and A. Ho introduced and studied the asymptotic norming property (ANP). Their main result is that a Banach space with the ANP has the RNP (= Radon-Nikodym property). This is of considerable interest because there exist separable spaces having the ANP (and therefore the RNP) which cannot be isomorphically embedded in separable duals [3, 4]. The proof of the theorem, however, is rather complicated. By using martingale theory we give a much shorter proof that directly extends Chatterji's methods [1].

Various formally different definitions of the ANP are given in [3]. There is no need here to state them and to show their equivalence. In fact our proof works for the formally weakest one, which we recall below. Let $X$ be a Banach space. We denote the unit ball $\{x: \|x\| \leqslant 1\}$ by $B(X)$ and the unit sphere $\{x: \|x\| = 1\}$ by $S(X)$. Our other terminology is standard. For notions unexplained we refer to [2]. A *norming set* for $X$ is a subset $\Phi$ of $B(X^*)$ such that $\|x\| = \sup\{\langle x, x^*\rangle: x^* \in \Phi\}$ for every $x \in X$. A sequence $(x_n) \subset S(X)$ is said to be *asymptotically normed* by $\Phi$ if for each $\varepsilon > 0$ there exist $x^* \in \Phi$ and $N \in \mathbf{N}$ such that $\langle x_n, x^*\rangle > 1 - \varepsilon$ whenever $n \geqslant N$.

DEFINITION. $X$ has the *asymptotic norming property* (ANP) if after some equivalent renorming there exists a subset $\Phi \subset B(X^*)$ with the following properties:

(i) $\Phi$ is a norming set for $X$;

(ii) for every sequence $(x_n) \subset S(X)$ which is asymptotically normed by $\Phi$ the set $\bigcap_{n=1}^{\infty} K_n$ is nonempty, where $K_n = \overline{\text{co}}\{x_i: i \geqslant n\}$ $(n = 1, 2, \ldots)$.

We now recall a basic result of Chatterji [1]. *A Banach space $X$ has the RNP iff every $X$-valued uniformly bounded martingale defined on* $[0, 1]$ *with Lebesgue measure $\lambda$ converges almost surely (a.s.) in norm.* The following lemma formulates a criterion for a.s. convergence of uniformly bounded martingales in dual spaces and will be useful later. We sketch a proof for completeness, although essentially this lemma is contained in [1].

LEMMA 1. *Let $X$ be a Banach space and let $(f_n, \Sigma_n, n \in \mathbf{N})$ be a uniformly bounded $X^*$-valued martingale on* $[0, 1]$. *Then the following holds.*

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(i) *There exists a uniformly bounded* $w^*$*-measurable function* $f\colon [0,1] \to X^*$ *with the property that for every* $x \in X$ *there is a* $\lambda$*-null set* $N_x$ *such that* $\lim_{n\to\infty}\langle x, f_n(\omega)\rangle = \langle x, f(\omega)\rangle$ *whenever* $\omega \notin N_x$.

(ii) $(f_n)$ *converges a.s. in norm iff there exists a* $\lambda$*-essentially separably valued* $f$ *satisfying* (i). *In this case* $f$ *is the a.s. limit of* $(f_n)$.

PROOF. For every $\omega \in [0,1]$ let $f(\omega)$ be a $w^*$-limit point of $(f_n(\omega))$. Since for each $x \in X$ the scalar-valued martingale $(\langle x, f_n\rangle, \Sigma_n, n \in \mathbf{N})$ converges a.s., and since, obviously, $\langle x, f\rangle$ is the only possible limit, (i) follows. For the proof of (ii) suppose there exists a $\lambda$-essentially separably valued $f$ as in (i). Let $W$ denote a separable subspace of $X^*$ that $\lambda$-essentially contains the ranges of $f$ and those of all $f_n$, and let $(x_n) \subset B(X)$ be a norming sequence for $W$. Since the functions $\langle x_n, f\rangle$ are all measurable it follows by a standard argument (cf. [**2**, Corollary II, 1.4]) that $f$ is measurable and therefore Bochner integrable. To show that $(f_n)$ converges a.s. it suffices to prove that $f_n = \mathcal{E}(f \mid \Sigma_n)$ $(n = 1, 2, \ldots)$ [**2**, Corollary V, 2.2 and Theorem V, 2.8]. Let $n \in \mathbf{N}$, $A \in \Sigma_n$ and $x \in X$ be arbitrary. Then we have

$$\left\langle x, \int_A f_n \, d\lambda \right\rangle = \int_A \langle x, f_n\rangle \, d\lambda = \lim_{m\to\infty} \int_A \langle x, f_m\rangle \, d\lambda$$
$$= \int_A \langle x, f\rangle \, d\lambda = \left\langle x, \int_A f \, d\lambda \right\rangle.$$

It follows that $\int_A f_n \, d\lambda = \int_A f \, d\lambda$, which means $\mathcal{E}(f \mid \Sigma_n) = f_n$. Clearly $f$ is the a.s. limit of $(f_n)$. The "only if" part of (ii) is obvious.

REMARK. The well-known fact that separable duals and reflexive spaces have the RNP is an immediate consequence of Lemma 1 and Chatterji's result.

The second tool we need is a lemma of Neveu [**5**, Lemma V, 2.9].

LEMMA 2. *For each* $m \in \mathbf{N}$ *let* $(f_n^{(m)}, \Sigma_n, n \in \mathbf{N})$ *be a real-valued submartingale on some probability space and assume the sequence* $(\sup_{m\in\mathbf{N}} f_n^{(m)+})_{n=1}^{\infty}$ *is* $L^1$*-bounded. Then*

(i) *each of the submartingales* $(f_n^{(m)})_{n=1}^{\infty}$ *converges a.s. to an integrable limit* $f^{(m)}$ $(m = 1, 2, \ldots)$, *and*

(ii) $\sup_{m\in\mathbf{N}} f_n^{(m)} \to \sup_{m\in\mathbf{N}} f^{(m)}$ *a.s. as* $n \to \infty$.

We are now ready for the main result [**3**, Theorem 1.8].

THEOREM. *A Banach space* $X$ *with the ANP has the RNP.*

PROOF. Since the RNP is separably determined and invariant for isomorphisms and since the ANP is clearly inherited by subspaces, we may assume that $X$ is separable and that the given norm on $X$ admits a set $\Phi \subset B(X^*)$ as in the Definition. We may further assume $\Phi$ is countable. ($\Phi$ contains a countable norming subset $\Phi'$ since $X$ is separable and every sequence in $S(X)$ asymptotically normed by $\Phi'$ is also asymptotically normed by $\Phi$.) As in [**3**] we now introduce on the linear subspace spanned by $\Phi$ a new norm, namely the gauge of $\operatorname{co}\{\Phi \cup -\Phi\}$. If $Y$ denotes the completion of $\operatorname{sp}\Phi$ for this new norm, then it is easy to see that $X$ is isometric to a subspace of $Y^*$ and $X$, with its given norm, satisfies (i) and (ii) in the

Definition with $\Phi$ equal to $B(Y)$. (Observe that a sequence in $S(X)$ asymptotically normed by $B(Y)$ has a subsequence asymptotically normed either by $\Phi$ or by $-\Phi$.) Notice also that $Y$ is separable.

Now let $(f_n, \Sigma_n, n \in \mathbf{N})$ be a uniformly bounded $X$-valued martingale defined on $[0, 1]$. What must be shown is that $(f_n)$ converges a.s. in norm. We regard $(f_n)$ as a $Y^*$-valued martingale and apply both lemmas. Let $f$ be as in Lemma 1(i), and let $(y_m)$ be a dense sequence in $B(Y)$. Lemma 2 applied with $f_n^{(m)} = \langle y_m, f_n \rangle$ $(n, m \in \mathbf{N})$ yields that

$$\sup_{m \in \mathbf{N}} \langle y_m, f_n \rangle = \|f_n\| \to \sup \langle y_m, f \rangle = \|f\| \quad \text{a.s.}$$

Let $N$ be a $\lambda$-null set such that

$$\langle y, f_n(\omega) \rangle \to \langle y, f(\omega) \rangle. \tag{1}$$

and

$$\|f_n(\omega)\| \to \|f(\omega)\| \tag{2}$$

hold for all $\omega \notin N$ and all $y \in \{y_1, y_2, \ldots\}$. Now fix $\omega \notin N$ and assume $\|f(\omega)\| = 1$ (the case $f(\omega) = 0$ is trivial). For any $\varepsilon > 0$ choose $y_m$ so that $\langle y_m, f(\omega) \rangle > 1 - \varepsilon$. Then also $\langle y_m, f_n(\omega) \rangle > 1 - \varepsilon$ for sufficiently large $n \in \mathbf{N}$. It follows now that the sequence $(f_n(\omega)/\|f_n(\omega)\|) \subset S(X)$ is asymptotically normed by $B(Y)$. Therefore

$$K := \bigcap_{n=1}^{\infty} K_n \neq \varnothing, \quad \text{where } K_n = \overline{\text{co}} \left\{ \frac{f_i(\omega)}{\|f_i(\omega)\|} : i \geqslant n \right\}.$$

It is now clear from (1) and (2) that $K = \{f(\omega)\}$. Since $K \subset X$ and $X$ is separable, we have now shown that $f$ is $\lambda$-essentially separably valued, and this concludes the proof by Lemma 1(ii).

## References

1. S. D. Chatterji, *Martingale convergence and the Radon-Nikodym theorem in Banach spaces*, Math. Scand. **22** (1968), 21–41.

2. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, no. 15, Amer. Math. Soc., Providence, R. I., 1977.

3. R. C. James and Aggie Ho, *The asymptotic-norming and Radon-Nikodym properties for Banach spaces*, Ark. Mat. **19** (1981), 53–70.

4. P. McCartney and R. O'Brien, *A separable Banach space with the Radon-Nikodym property which is not isomorphic to a subspace of a separable dual*, Proc. Amer. Math. Soc. **78** (1980), 40–42.

5. J. Neveu, *Discrete-parameter martingales*, North-Holland, Amsterdam, 1975.

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