THE ASYMPTOTIC NORMING PROPERTY
AND MARTINGALE CONVERGENCE

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Abstract. A martingale proof is given of the result of R. G. James and A. Ho in [3] that the asymptotic norming property implies the Radon-Nikodym property.

In [3] R. C. James and A. Ho introduced and studied the asymptotic norming property (ANP). Their main result is that a Banach space with the ANP has the RNP (= Radon-Nikodym property). This is of considerable interest because there exist separable spaces having the ANP (and therefore the RNP) which cannot be isomorphically embedded in separable duals [3, 4]. The proof of the theorem, however, is rather complicated. By using martingale theory we give a much shorter proof that directly extends Chatterji's methods [1].

Various formally different definitions of the ANP are given in [3]. There is no need here to state them and to show their equivalence. In fact our proof works for the formally weakest one, which we recall below. Let X be a Banach space. We denote the unit ball \( \{ x : \|x\| \leq 1 \} \) by \( B(X) \) and the unit sphere \( \{ x : \|x\| = 1 \} \) by \( S(X) \). Our other terminology is standard. For notions unexplained we refer to [2]. A norming set for \( Y \) is a subset \( \Phi \) of \( B(X^*) \) such that \( \|x\| = \sup \{ \langle x, x^* \rangle : x^* \in \Phi \} \) for every \( x \in X \). A sequence \( (x_n) \subset S(X) \) is said to be asymptotically normed by \( \Phi \) if for each \( \varepsilon > 0 \) there exist \( x^* \in \Phi \) and \( N \in \mathbb{N} \) such that \( \langle x_n, x^* \rangle > 1 - \varepsilon \) whenever \( n \geq N \).

Definition. \( X \) has the asymptotic norming property (ANP) if after some equivalent renorming there exists a subset \( \Phi \subset B(X^*) \) with the following properties:

(i) \( \Phi \) is a norming set for \( X \);

(ii) for every sequence \( (x_n) \subset S(X) \) which is asymptotically normed by \( \Phi \) the set \( \bigcap_{n=1}^{\infty} K_n \) is nonempty, where \( K_n = \text{co} \{ x_i : i \geq n \} \) \((n = 1, 2, \ldots)\).

We now recall a basic result of Chatterji [1]. A Banach space \( X \) has the RNP iff every \( X \)-valued uniformly bounded martingale defined on \([0,1]\) with Lebesgue measure \( \lambda \) converges almost surely (a.s.) in norm. The following lemma formulates a criterion for a.s. convergence of uniformly bounded martingales in dual spaces and will be useful later. We sketch a proof for completeness, although essentially this lemma is contained in [1].

Lemma 1. Let \( X \) be a Banach space and let \( (f_n, \Sigma_n, n \in \mathbb{N}) \) be a uniformly bounded \( X^* \)-valued martingale on \([0,1]\). Then the following holds.

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(i) There exists a uniformly bounded w*-measurable function \( f : [0, 1] \to X^* \) with the property that for every \( x \in X \) there is a \( \lambda \)-null set \( N_x \) such that \( \lim_{n \to \infty} \langle x, f_n(\omega) \rangle = \langle x, f(\omega) \rangle \) whenever \( \omega \notin N_x \).

(ii) \( (f_n) \) converges a.s. in norm iff there exists a \( \lambda \)-essentially separably valued \( f \) satisfying (i). In this case \( f \) is the a.s. limit of \( (f_n) \).

**Proof.** For every \( \omega \in [0, 1] \) let \( f(\omega) \) be a w*-limit point of \( (f_n(\omega)) \). Since for each \( x \in X \) the scalar-valued martingale \( (\langle x, f_n \rangle, \Sigma_n, n \in \mathbb{N}) \) converges a.s., and since, obviously, \( \langle x, f \rangle \) is the only possible limit, (i) follows. For the proof of (ii) suppose there exists a \( \lambda \)-essentially separably valued \( f \) as in (i). Let \( W \) denote a separable subspace of \( X^* \) that \( \lambda \)-essentially contains the ranges of \( f \) and those of all \( f_n \), and let \( (x_n) \subset B(X) \) be a norming sequence for \( W \). Since the functions \( \langle x_n, f \rangle \) are all measurable it follows by a standard argument (cf. \[2, Corollary II, 1.4\]) that \( f \) is measurable and therefore Bochner integrable. To show that \( (f_n) \) converges a.s. it suffices to prove that \( f_n = \delta(\Sigma_n) (n = 1, 2, \ldots) \) \[2, Corollary V, 2.2 and Theorem V, 2.8\]. Let \( n \in \mathbb{N}, A \in \Sigma_n \) and \( x \in X \) be arbitrary. Then we have

\[
\left\langle x, \int_A f_n \, d\lambda \right\rangle = \int_A \left\langle x, f_n \right\rangle \, d\lambda = \lim_{m \to \infty} \int_A \left\langle x, f_m \right\rangle \, d\lambda = \int_A \left\langle x, f \right\rangle \, d\lambda = \left\langle x, \int_A f \, d\lambda \right\rangle.
\]

It follows that \( \int_A f_n \, d\lambda = \int_A f \, d\lambda \), which means \( \delta(f | \Sigma_n) = f_n \). Clearly \( f \) is the a.s. limit of \( (f_n) \). The “only if” part of (ii) is obvious.

**Remark.** The well-known fact that separable duals and reflexive spaces have the RNP is an immediate consequence of Lemma 1 and Chatterji’s result.

The second tool we need is a lemma of Neveu \[5, Lemma V, 2.9\].

**Lemma 2.** For each \( m \in \mathbb{N} \) let \( (f_n^{(m)}, \Sigma_n, n \in \mathbb{N}) \) be a real-valued submartingale on some probability space and assume the sequence \( (\sup_{m \in \mathbb{N}} f_n^{(m)+})_{n=1}^{\infty} \) is \( L^1 \)-bounded. Then

(i) each of the submartingales \( (f_n^{(m)})_{n=1}^{\infty} \) converges a.s. to an integrable limit \( f^{(m)} \) (\( m = 1, 2, \ldots \)), and

(ii) \( \sup_{m \in \mathbb{N}} f_n^{(m)} \to \sup_{m \in \mathbb{N}} f^{(m)} \) a.s. as \( n \to \infty \).

We are now ready for the main result \[3, Theorem 1.8\].

**Theorem.** A Banach space \( X \) with the ANP has the RNP.

**Proof.** Since the RNP is separably determined and invariant for isomorphisms and since the ANP is clearly inherited by subspaces, we may assume that \( X \) is separable and that the given norm on \( X \) admits a set \( \Phi \subset B(X^*) \) as in the Definition. We may further assume \( \Phi \) is countable. (\( \Phi \) contains a countable norming subset \( \Phi' \) since \( X \) is separable and every sequence in \( S(X) \) asymptotically normed by \( \Phi' \) is also asymptotically normed by \( \Phi \).) As in \[3\] we now introduce on the linear subspace spanned by \( \Phi \) a new norm, namely the gauge of \( \text{co}(\Phi \cup -\Phi) \). If \( Y \) denotes the completion of \( \text{sp} \Phi \) for this new norm, then it is easy to see that \( X \) is isometric to a subspace of \( Y^* \) and \( X \), with its given norm, satisfies (i) and (ii) in the
Definition with $\Phi$ equal to $B(Y)$. (Observe that a sequence in $S(X)$ asymptotically normed by $B(Y)$ has a subsequence asymptotically normed either by $\Phi$ or by $-\Phi$.) Notice also that $Y$ is separable.

Now let $(f_n, \Sigma_n, n \in \mathbb{N})$ be a uniformly bounded $X$-valued martingale defined on $[0, 1]$. What must be shown is that $(f_n)$ converges a.s. in norm. We regard $(f_n)$ as a $Y^*$-valued martingale and apply both lemmas. Let $f$ be as in Lemma 1(i), and let $(y_m)$ be a dense sequence in $B(Y)$. Lemma 2 applied with $f_n^{(m)} = \langle y_m, f_n \rangle (n, m \in \mathbb{N})$ yields that

$$\sup_{m \in \mathbb{N}} \langle y_m, f_n \rangle = ||f_n|| \to \sup \langle y_m, f \rangle = ||f|| \quad \text{a.s.}$$

Let $N$ be a $\lambda$-null set such that

(1) $\langle y, f_n(\omega) \rangle \to \langle y, f(\omega) \rangle$.

and

(2) $||f_n(\omega)|| \to ||f(\omega)||$

hold for all $\omega \notin N$ and all $y \in \{y_1, y_2, \ldots\}$. Now fix $\omega \notin N$ and assume $||f(\omega)|| = 1$ (the case $f(\omega) = 0$ is trivial). For any $\epsilon > 0$ choose $y_m$ so that $\langle y_m, f(\omega) \rangle > 1 - \epsilon$. Then also $\langle y_m, f_n(\omega) \rangle > 1 - \epsilon$ for sufficiently large $n \in \mathbb{N}$. It follows now that the sequence $(f_n(\omega)/||f_n(\omega)||) \subset S(X)$ is asymptotically normed by $B(Y)$. Therefore

$$K := \bigcap_{n=1}^{\infty} K_n \neq \varnothing, \text{ where } K_n = \text{co} \left\{ \frac{f_i(\omega)}{||f_i(\omega)||} : i \geq n \right\}.$$

It is now clear from (1) and (2) that $K = \{f(\omega)\}$. Since $K \subset X$ and $X$ is separable, we have now shown that $f$ is $\lambda$-essentially separably valued, and this concludes the proof by Lemma 1(ii).

REFERENCES


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