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NEGLIGIBLE SETS OF RADON MEASURES

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ABSTRACT. Let m be a Radon measure on a Hausdorff topological space X. Corresponding to three kinds of outer measures, three kinds of m-negligible sets are considered. The main theorem states that in a metacompact space X each locally m-negligible set is m-negligible.

For a Radon measure in a Hausdorff topological space X we distinguish three kinds of negligible sets corresponding to three kinds of outer measures: strictly negligible, negligible and locally negligible sets, respectively. It is shown that two classes of negligible sets coincide iff the corresponding outer measures coincide. Further, each locally negligible set is strictly negligible if X is Lindelöf. We give two examples that both in locally compact separable spaces and in locally compact metrizable spaces a negligible set need not be strictly negligible.

It is an unsolved problem whether a locally negligible set is always negligible, or equivalently, whether the corresponding outer measures are always the same. In this latter form the problem is due to Schwartz [**5**, p. 17]. The main purpose of this note is to prove that in a metacompact space, and hence in a paracompact space, each locally negligible set is negligible. It will turn out that this is even true for compact inner regular Borel measures which are only locally σ-finite. At least in this slightly more general situation it can be shown that the assumption "metacompact" cannot be replaced by "locally compact".

By the compact inner regularity of Radon measures the necessity to distinguish several kinds of negligible sets arises with the infinity of the measures. That is, for finite, compact inner regular Borel measures the three kinds of negligible sets coincide. In the case of finite Borel measures (not necessarily compact inner regular) there are at most two kinds of negligible sets. The class of topological spaces in which these two kinds coincide for each finite or each finite regular Borel measure was considered by Gardner in [**3**] (see also [**4**]).

Throughout this paper X is a Hausdorff topological space. $\mathfrak{G}(X)$, $\mathfrak{F}(X)$ and $\mathfrak{K}(X)$ denote, respectively, the families of all open, closed and compact subsets of X. The Borel σ-algebra is denoted by $\mathfrak{B}(X)$. A Borel measure in X, i.e. a nonnegative, countably additive set function on $\mathfrak{B}(X)$, is called locally finite if each point in X has a neighbourhood of finite measure.

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Let $\mathfrak{R}$ be the family of all locally finite Borel measures m in X satisfying the following two properties:

$$(1) \qquad m(K) = \inf\{m(G): \mathfrak{G}(X) \ni G \supset K\} \quad \text{for } K \in \mathfrak{K}(X);$$

$$(2) \qquad m(B) = \sup\{m(K): \mathfrak{K}(X) \ni K \subset B\} \quad \text{for } B \in \mathfrak{B}(X).$$

Since m is locally finite (2) implies (1). Further let $\tilde{\mathfrak{R}}$ be the family of all locally finite Borel measures M in X satisfying:

$$(\tilde{1}) \qquad M(G) = \sup\{M(K): \mathfrak{K}(X) \ni K \subset G\} \quad \text{for } G \in \mathfrak{G}(X);$$

$$(\tilde{2}) \qquad M(B) = \inf\{M(G): \mathfrak{G}(X) \ni G \supset B\} \quad \text{for } B \in \mathfrak{B}(X).$$

As was shown by L. Schwartz (see [**5**, pp. 13–15]) there is a natural bijection $\psi: \mathfrak{R} \to \tilde{\mathfrak{R}}$. In fact, for each $m \in \mathfrak{R}$ there exists the smallest measure $M = \psi(m) \in \tilde{\mathfrak{R}}$ dominating m; M is given by

$$M(B) = \inf\{m(G): B \subset G \in \mathfrak{G}(X)\} \quad \text{for } B \in \mathfrak{B}(X).$$

Conversely, for each $M \in \tilde{\mathfrak{R}}$ there exists the largest measure $m = \psi^{-1}(M) \in \mathfrak{R}$ dominated by M; m is given by

$$m(B) = \sup\{M(K): B \supset K \in \mathfrak{K}(X)\} \quad \text{for } B \in \mathfrak{B}(X).$$

It is easy to see that

$$(3) \qquad m(B) = M(B) \quad \text{for } B \in \mathfrak{B}(X) \text{ with } M(B) < \infty.$$

DEFINITION 1. A Borel measure m in X is called a *Radon measure* if m belongs to the class $\mathfrak{R}$. M always denotes the measure $\psi(m) \in \tilde{\mathfrak{R}}$.

If μ is a nonnegative measure on an arbitrary measurable space $(Y, \mathfrak{A})$, the essential outer measure $\mu^{\bar{*}}$ associated with μ is defined by

$$\mu^{\bar{*}}(A) = \sup\{\mu^*(A \cap B): \mu^*(B) < \infty\} \quad \text{for each } A \subset Y.$$

Because of (2) and (3) the measure $m = \psi^{-1}(M)$ can also be characterized by $m = M^{\bar{*}}|\mathfrak{B}(X)$. Beyond this the (essential) outer measures satisfy the following relations.

LEMMA.

$$m^{\bar{*}} \underset{(\alpha)}{=} M^{\bar{*}} \underset{(\beta)}{\leqslant} m^* \underset{(\gamma)}{\leqslant} M^*.$$

PROOF. (γ) is a trivial consequence of the definitions. By (3)

$$M^*(C) = m^*(C) \quad \text{for } C \subset X \text{ with } M^*(C) < \infty,$$

which implies (β). Equation (α) is obtained as follows (cf. [**5**, Lemma I.1]):

$$\begin{aligned} M^{\bar{*}}(A) &= \sup\{m^*(A \cap K): K \in \mathfrak{K}(X)\} \quad \text{(by (3) and (2))} \\ &= m^{\bar{*}}(A) \quad \text{(by (2))}. \quad \square \end{aligned}$$

According to the preceding lemma we distinguish three kinds of negligible sets.

DEFINITION 2. Let m be a Radon measure in X. A subset A of X is called

(i) strictly m-negligible if $M^*(A) = 0$,

(ii) m-negligible if $m^*(A) = 0$,

(iii) locally m-negligible if $m^{\bar{*}}(A) = 0$.

Thus a subset $A \subset X$ is strictly m-negligible iff $\inf\{m(G): A \subset G \in \mathfrak{G}(X)\} = 0$. Further, $A \subset X$ is locally m-negligible iff each point in X has a neighbourhood U with $m^*(A \cap U) = 0$. The comparison with the notations of Schwartz and Bourbaki yields:

	Schwartz	Bourbaki
$M^*(A) = 0$:	strictly m-negligible	m-negligible
$m^*(A) = 0$:	–	–
$m^{\bar{*}}(A) = 0$:	m-negligible	locally m-negligible

The question arises when two of the three classes of negligible sets do coincide. A first answer is given by

PROPOSITION 1. *Let m be a Radon measure in X.*
(a) *Each m-negligible set is strictly m-negligible iff* $m^* = M^*$.
(b) *Each locally m-negligible set is m-negligible iff* $m^{\bar{*}} = m^*$.

PROOF. To prove the nontrivial implications let A be an arbitrary subset of X.
(a) We may assume $A \in \mathfrak{B}(X)$ and $m(A) < \infty$. Hence there are compact sets $K_n \subset A$, $n \in \mathbf{N}$, such that $m(N) = 0$ for $N = A - \bigcup_{n \in \mathbf{N}} K_n$.
Since $M(K_n) < \infty$ for each $n \in \mathbf{N}$ the equation $M(A) = m(A)$ follows from (3) and $M(N) = 0$.
(b) 1. Now we may assume $m^{\bar{*}}(A) < \infty$. By (2) and the definition of $m^{\bar{*}}$ there are compact sets K_n, $n \in \mathbf{N}$, such that

$$m^{\bar{*}}(A) = m^*(K_\sigma \cap A) \quad \text{for } K_\sigma = \bigcup_{n \in \mathbf{N}} K_n.$$

As will be shown in the second part, $N := A - K_\sigma$ is locally m-negligible and, hence, m-negligible. Therefore we have

$$m^*(A) \leqslant m^*(K_\sigma \cap A) + m^*(N) = m^{\bar{*}}(A) \leqslant m^*(A).$$

2. Assume $m^{\bar{*}}(N) > 0$. Then we can find a compact set K such that $\varepsilon := m^*(K \cap N) > 0$. Let G_n be open sets satisfying $G_n \supset K_n$ and $m(G_n - K_n) < \varepsilon \cdot 2^{-n}$ for each $n \in \mathbf{N}$. It follows that $K' := K \setminus (\bigcup_{n \in \mathbf{N}} G_n)$ is compact with $K' \cap K_\sigma = \varnothing$ and $K \cap N \subset (K' \cap N) \cup \bigcup_{n \in \mathbf{N}} (G_n - K_n)$, hence $m^*(K' \cap N) > m^*(K \cap N) - \varepsilon = 0$. This implies

$$m^*((K_\sigma \cup K') \cap A) = m^*(K_\sigma \cap A) + m^*(K' \cap A) > m^{\bar{*}}(A),$$

which is a contradiction. □
Obviously if M is σ-finite then $m^{\bar{*}} = m^* = M^*$. From this we obtain

PROPOSITION 2. *Let X be Lindelöf and let m be a Radon measure in X. Then each locally m-negligible set is strictly m-negligible.*

PROOF. The assertion follows from the fact that a locally finite Borel measure in X is σ-finite. □
Example 1 shows there is a locally compact, metrizable space with a Radon measure m such that $m^* \neq M^*$. This may also happen in a locally compact, separable space (see Example 2), whereas $m^* = M^*$ holds on a metacompact separable space X since X is Lindelöf.

It is an unsolved problem whether a locally m-negligible set is always m-negligible. According to Proposition 1 this is equivalent to the problem of Schwartz in [**5**, p. 17], where he asks for a Radon measure with $\overline{m^*} \neq m^*$. Equality holds in a relatively wide class of topological spaces:

THEOREM. *Let m be a Radon measure on a metacompact space X. Then each locally m-negligible set N is m-negligible.*

PROOF. Let $\mathfrak{G}_0$ be an open point finite refinement of $\{U \in \mathfrak{G}(X): m(U) < \infty\}$. Since $m|\mathfrak{G}(X) = M|\mathfrak{G}(X)$ it follows from (3) and $(\tilde{2})$ that for each $U \in \mathfrak{G}_0$ there exists a sequence $(G_U^n)_{n \in \mathbf{N}}$ in $\mathfrak{G}(X)$ such that:

(i) $N \cap U \subset G_U^n \subset U$ for each $n \in \mathbf{N}$,

(ii) $G_U^{n+1} \subset G_U^n$ for each $n \in \mathbf{N}$,

(iii) $m(\bigcap_{n \in \mathbf{N}} G_U^n) = 0$.

Now for $A := \bigcap_{n \in \mathbf{N}} \bigcup_{U \in \mathfrak{G}_0} G_U^n$ we get:

(a) $A \in \mathfrak{B}(X)$: Trivial.

(b) $A \supset N$: Because of $\bigcup \mathfrak{G}_0 = X$ and (i).

(c) $m(A) = 0$: It is sufficient to show $m(K) = 0$ for each compact set $K \subset A$. For each $n \in \mathbf{N}$ we have $K \subset \bigcup_{U \in \mathfrak{G}_0} G_U^n$. Hence there exists a countable subcollection $\mathfrak{A}_K$ of $\mathfrak{G}_0$ such that

$$K \subset \bigcap_{n \in \mathbf{N}} \bigcup_{U \in \mathfrak{A}_K} G_U^n.$$

Since each point of X lies in only finitely many members of $\mathfrak{G}_0$, we obtain from (i) and (ii):

$$\bigcap_{n \in \mathbf{N}} \bigcup_{U \in \mathfrak{A}_K} G_U^n = \bigcup_{U \in \mathfrak{A}_K} \bigcap_{n \in \mathbf{N}} G_U^n.$$

Together with (iii) it follows that $m(K) = 0$. □

REMARK. It is easy to see that the Theorem remains true if m is only a locally σ-finite (instead of locally finite) Borel measure satisfying (2). In the locally σ-finite situation, however, the Theorem is false if "metacompact" is replaced by "locally compact" (see Example 3). The author does not know if the Theorem is true for Radon measures on locally compact spaces.

EXAMPLE 1. Let X be the product of a discrete space I and $Y = [0, 1]$, where Y is provided with the usual topology. By λ we denote the Lebesgue measure on Y. For each $B \in \mathfrak{B}(X)$, let

$$m(B) = \sum_{i \in I} \lambda(\{y \in Y: (i, y) \in B\}).$$

Clearly X is a metrizable locally compact space and m is a Radon measure in X. The set $A = I \times \{0\}$ is m-negligible but not strictly m-negligible if I is uncountable.

EXAMPLE 2. On $X = \bigcup_{1 \leqslant k \leqslant \infty} X_k$ with $X_k = \{0, 1\}^k$ for $k \in \mathbf{N}$ and $X_\infty = \{0, 1\}^{\mathbf{N}}$, we define a topology as follows: the points $x \in X - X_\infty$ are isolated and a neighbourhood base at $x \in X_\infty$ is given by the sets

$$U_n(x) = \{x\} \cup \{(\pi_1(x), \ldots, \pi_j(x)): n \leqslant j < \infty\}, \qquad n \in \mathbf{N},$$

where π_i denotes the ith projection of X_k on $\{0, 1\}$ for $i \leqslant k \leqslant \infty$. It is easy to see that X is a separable locally compact Hausdorff space. Further,

$$m = \sum_{1 \leqslant k < \infty} 2^{-k} \cdot \sum_{x \in X_k} \delta_x$$

is a σ-finite Radon measure on X with the m-negligible set X_∞. X_∞ is not strictly m-negligible, which can be seen as follows: Assume there exists $G \in \mathfrak{G}(X)$ with $X_\infty \subset G$ and $m(G) < \infty$. For $n \in \mathbf{N}$ and $(z_1, \ldots, z_n) \in X_n$ let

$$A(z_1, \ldots, z_n) = \left\{ x \in \bigcup_{n \leqslant k < \infty} X_k \colon \pi_i(x) = z_i \text{ for } 1 \leqslant i \leqslant n \right\}.$$

Since $A(z_1, \ldots, z_n) \cap X_k$, $n \leqslant k < \infty$, contains 2^{k-n} points, it follows that $m(A(z_1, \ldots, z_n)) = \infty$. Hence we can choose a sequence $(z_1, \ldots, z_{n_j})_{j \in \mathbf{N}}$ in $X - X_\infty$ such that $(z_1, \ldots, z_{n_{j+1}}) \in A(z_1, \ldots, z_{n_j}) \setminus G$ for each $j \in \mathbf{N}$. From this and the definition of the topology it follows that $z = (z_i)_{i \in \mathbf{N}} \in X_\infty$ and $z \notin G$, which is a contradiction to $X_\infty \subset G$.

REMARK. A similar example is given in [**1**, Chapter IV, p. 234, exercise 5] using a category argument.

EXAMPLE 3. Let X be the space of all countable ordinals with the order topology. Let A be a subset of X such that neither A nor $X - A$ contains an uncountable closed subset. Then X is a locally compact Hausdorff space and $m|\mathfrak{B}(X)$ with

$$m(B) = \sup\{|B \cap T| \colon T \subset A \text{ finite}\} \quad \text{for } B \in \mathfrak{B}(X)$$

is a locally σ-finite Borel measure satisfying (2). Obviously $N = X - A$ is locally m-negligible but $m^*(N) = \infty$. In fact, if $B \in \mathfrak{B}(X)$ contains N, then B contains an uncountable closed subset, since otherwise $X - B$ and, hence, A would contain an uncountable closed subset (cf. [**2**]). Now $F \cap A$ is uncountable for each uncountable closed subset F. This implies $m(B) = \infty$.

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REFERENCES

1. N. Bourbaki, *Intégration*, Chapitres I–IV, 2ième ed., Hermann, Paris, 1965.

2. J. Dieudonné, *Un exemple d'espace normale non susceptible d'une structure uniforme d'espace complet*, C. R. Acad. Sci. Paris **209** (1939), 145–147.

3. R. J. Gardner, *The regularity of Borel measures and Borel measure-compactness*, Proc. London Math. Soc. (3) **30** (1975), 95–113.

4. R. A. Johnson, *Another Borel measure-compact space which is not weakly Borel measure-complete*, J. London Math. Soc. (2) **21** (1980), 263–264.

5. L. Schwartz, *Radon measures on arbitrary topological spaces and cylindrical measures*, Oxford Univ. Press, London, 1973.

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